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Special geometry of euclidean supersymmetry 1. Vector multiplets*

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Abstract: We construct the general action for abelian vector multiplets in rigid 4-dimensional euclidean (instead of minkowskian) $\mathcal{N} = 2$ supersymmetry, i.e., over space-times with a positive definite instead of a lorentzian metric. The target manifolds for the scalar fields turn out to be para-complex manifolds endowed with a particular kind of special geometry, which we call affine special para-Kähler geometry. We give a precise definition and develop the mathematical theory of such manifolds. The relation to the affine special Kähler manifolds appearing in minkowskian $\mathcal{N} = 2$ supersymmetry is discussed. Starting from the general five-dimensional vector multiplet action we consider dimensional reduction over time and space in parallel, providing a dictionary between the resulting euclidean and minkowskian theories. Then we reanalyze supersymmetry in four dimensions and find that any (para-)holomorphic prepotential defines a supersymmetric lagrangian, provided that we add a specific four-fermion term, which cannot be obtained by dimensional reduction. We show that the euclidean action and supersymmetry transformations, when written in terms of para-holomorphic coordinates, take exactly the same form as their minkowskian counterparts. The appearance of a para-complex and complex structure in the euclidean and minkowskian theory, respectively, is traced back to properties of the underlying R-symmetry groups. Finally, we indicate how our work will be extended to other types of multiplets and to supergravity in the future and explain the relevance of this project for the study of instantons, solitons and cosmological solutions in supergravity and M-theory.

Keywords: Sigma Models, Space-Time Symmetries, Extended Supersymmetry, Supersymmetric Effective Theories.

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1. Introduction, summary, conclusions and outlook

1.1 Introduction

Most of the knowledge about the non-perturbative properties of string theory and M-theory relies on dualities, which re-interpret the strong coupling behaviour of particular limits of M-theory in terms of a dual, weakly coupled description [1]. Although string dualities have passed various highly non-trivial tests, they still have the status of conjectures. Moreover,

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they address non-perturbative physics only indirectly. Therefore the further development of non-perturbative methods in string and M-theory is desirable. In particular one would like to have the analogue of the instanton calculus used in gauge theories. The first important step in this direction was the discovery of the D-instanton [2] in IIB string theory. It was then realized that instanton effects in M-theory compactifications correspond to euclidean wrappings of p-branes on (p+1)-cycles of the internal manifold [3] (see also [4] for a review and more references). Similar to string and M-theory solitons, these instantons can be described in terms of the low energy effective supergravity action. In this formulation they are instanton solutions, i.e., solutions of the euclidean theory which have a finite action. The IIB supergravity solution corresponding to the D-instanton was found in [5]. There are other solutions corresponding to instantons in Calabi-Yau compactifications in type-II string theory [6, 7] and in 11-dimensional M-theory [8].

Let us outline the problems which we will address in this paper. One particular question arising in the context of D-instantons and their generalizations is how to define the euclidean supergravity action. In their supergravity description of the D-instanton [5] G.W. Gibbons, M.B. Green and M.J. Perry invoked an elegant but somewhat mysterious rule, which requires to replace factors of i in the lagrangian and in the supersymmetry rules by a formal factor e, satisfying $\overline{e} = -e$ and $e^2 = 1$. (The formal factor e should be interpreted as an imaginary unit in the algebra of para-complex numbers.) The same euclidean lagrangian was obtained in [9] by first Hodge-dualizing the IIB-axion into a tensor field, then performing a Wick rotation, and dualizing afterwards the tensor field back to a scalar. Due to properties of the Hodge star operator in euclidean signature, the euclidean action with a tensor field is definite, i.e., bounded from above (or from below, depending on the choice of overall sign), while the dual action with a scalar field is indefinite.[1] The D-instanton can be described in terms of both actions, but the use of the indefinite action makes it particularly simple to find explicit solutions [5]. Of course, instanton corrections should be computed using the definite version of the action, as was recently emphasized by [7] in the context of type-II Calabi-Yau compactifications.

String and M-theory instantons and solitons are related to one another by dimensional reduction and T-duality transformations. In fact, instanton solutions can be used to generate solitons by 'dimensional oxidation' [10], see [11] for a review. In this approach one compactifies all world-volume directions of the brane, including time. The dimensional reduction over time has the effect that the metric of the scalar manifold becomes indefinite, see formulae (1.1), (1.4) below. Instanton solutions are given by harmonic maps from the transverse space of the brane into totally geodesic and totally isotropic submanifolds (such as null geodesics) of the scalar manifold. They can be re-expressed in terms of the quantities of the original higher-dimensional theory to obtain the corresponding soliton. This technique goes by the name of dimensional oxidation. Obviously one needs to know explicitly how the fields of the lower-dimensional theory are related to those of the higher dimensional one. This is one of the reasons for studying euclidean theories using dimensional reduction over time.

[1] Notice that an indefinite target metric for the scalar fields implies an indefinite action. The converse is not true. Equivalently, a definite action implies a (positive or negative) definite target metric.

Finally, scalar manifolds of the same type as in euclidean theories also occur in non-standard Minkowski signature supergravity and string theories, which are obtained by T-duality transformations over time. Particular examples are the type-II* string theories introduced in [12]. These theories have interesting cosmological solutions, which are supersymmetric and asymptotically de Sitter.[2] In [13] it was shown that asymptotic de Sitter solutions of non-standard gauged supergravities in 4 and 5 dimensions can be obtained by massive time-like T-duality transformations from instanton solutions of ungauged supergravity. These solutions are believed to be related to type-II* supergravity theory by dimensional reduction.

The geometrical structures of the scalar manifolds appearing in the above examples deserve a closer analysis. For example, in [5] the $i \to e$ substitution rule gives the desired result, a description of the D-instanton in terms of supergravity, but it leads to the immediate question: *what is the geometrical meaning of the $i \to e$ substitution rule?* More generally, we can ask what characterizes the geometries corresponding to supersymmetric theories obtained by time-like dimensional reduction and time-like T-duality. In theories with 32 or 16 supercharges the scalar manifolds are symmetric spaces, which are fixed uniquely by the matter content. These manifolds can be found case by case [14, 15]. The scalar geometry becomes richer when reducing the number of supersymmetries. In this paper we will consider theories which have $\mathcal{N} = 2$ supersymmetry, i.e., eight real supercharges.[3] The scalar manifolds of $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories are not fixed by the matter content. But whereas the scalar manifolds of $\mathcal{N} = 1$ theories are (for minkowskian space-time) generic Kähler manifolds, those of $\mathcal{N} = 2$ are subject to more specific conditions. The resulting geometries are therefore called *special geometries*. The precise type of geometry depends on (i) whether supersymmetry is a rigid or local symmetry, (ii) the type of supermultiplet and (iii) the number of space-time dimensions. We refer to [16] for an overview of special geometries and their mutual relations. In this context we can now rephrase our above question as follows: *what are the special geometries of $\mathcal{N} = 2$ theories with euclidean signature?* The present paper is the first in a series which will answer this question. For technical simplicity we will start with rigid supersymmetry and develop the geometry of euclidean $\mathcal{N} = 2$ vector multiplets in 4 dimensions. In subsequent work we will extend this to other multiplets and dimensions as well as to supergravity. The results will be used to study instantons and solitons in string and M-theory compactifications.

There are two methods which can be applied to find a euclidean action. The first method is the dimensional reduction of a higher-dimensional minkowskian action over time. This has been used in [17, 18] for 4-dimensional $\mathcal{N} = 2$ Yang-Mills theory and in [14] for euclidean supergravity theories. The second approach is to continue the 4-dimensional minkowskian theory analytically to imaginary time. Here there are two different versions. The first version was used in [19, 20] to construct 4-dimensional euclidean gauge theories. More recently, this construction has been generalized to a continuous Wick rotation

[2] However, as discussed in [12], unitarity and stability of these theories are unclear.

[3] We are counting in units of 4-dimensional minkowskian supersymmetry. Note that some of the theories we refer to as $\mathcal{N} = 2$ for terminological convenience are actually the minimal supersymmetric theories in their dimension and signature.

by [21, 22, 23]. As in the treatment of IIB supergravity in [5], one obtains an indefinite supersymmetric euclidean action. Moreover, the action obtained by the continuous Wick rotation agrees with the one obtained by dimensional reduction over time [17, 18]. The second method using analytic continuation is based on the Osterwalder-Schrader formulation of euclidean field theories, and has been studied for 4-dimensional supersymmetric theories in [24]. In this approach one uses reality constraints which differ from those in the continuous Wick rotation, as for fermions hermitean conjugation is combined with euclidean time reflection. Moreover, the euclidean action is definite. The euclidean actions obtained in the Osterwalder-Schrader are certainly the correct actions to be used in the path integral quantization and in lattice studies of supersymmetric field theories. However, they are not suitable for the applications we are interested in. To find generalizations of D-instantons we need to study the euclidean versions of multiplets which fundamentally are vector-tensor multiplets rather than vector multiplets, in analogy to the situation in IIB string theory discussed above. And to construct solitons by dimensional oxidation, we need euclidean theories which are obtainable from higher-dimensional theories by dimensional reduction over time. The relation between the two types of euclidean continuations of supersymmetric actions has been discussed in [22, 25], and we plan to further analyze it in a future publication.

In this paper we construct the action of 4-dimensional euclidean vector multiplets by dimensional reduction.[4] We consider the reduction of the general 5-dimensional action for abelian vector multiplets over a time-like and a space-like dimension in parallel. This way we obtain a dictionary between the 4-dimensional theories in both signatures, and we can compare with the results [26, 27] for vector multiplets in Minkowski signature. The action obtained by dimensional reduction is not the most general one. In 5 dimensions the presence of a Chern-Simons term forces the prepotential, which encodes the whole lagrangian, to be a cubic polynomial. This constraint is absent in 4 dimensions, where the only condition is that the prepotential is a (para-)holomorphic function. The general lagrangian is obtained by reanalyzing the supersymmetry transformations for a general (para-)holomorphic prepotential, with the result that a particular four-fermion term has to be added. In the minkowskian case the resulting lagrangian agrees with the general vector multiplet lagrangian of [26, 27].[5]

Our action for 4-dimensional euclidean vector multiplets is real but indefinite. Similar to the case of type-IIB supergravity discussed above [5, 9], a dual action, which is both real and definite can be obtained by Hodge-dualizing the $\mathcal{N} = 2$ vector multiplets into $\mathcal{N} = 2$ vector-tensor multiplets [28]. As is known from [29], this dualization is only possible for a restricted class of vector multiplet couplings. However, for the applications we have in mind it is guaranteed that the dualization is possible, because the vector multiplets relevant for string instantons are those which have been obtained by dualizing vector-tensor multiplets. The most important case is heterotic $\mathcal{N} = 2$ compactifications, where the dilaton sits in a vector-tensor multiplet, which is then dualized into a vector multiplet [72].

[4] A generalization of the continuous Wick rotation of [22] will be discussed in a separate paper.

[5] We only consider lagrangians which contain at most second derivatives of the fields and no terms of order higher than four in the fermions.

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The dilaton controls the quantum corrections to the vector multiplet part of the effective action. So far only perturbative corrections have been computed directly, but the instanton corrections are predicted by the duality to type-II compactifications on Calabi-Yau three-folds [31]. There are also heterotic compactifications with more than one vector-tensor multiplet, namely toroidal compactifications of six-dimensional string vacua with tensor multiplets [32, 33].

In the next subsection we will discuss the effects of dimensional reduction over time in a simple, but instructive example.

1.2 Summary and conclusions

One way to obtain a d-dimensional euclidean action is by dimensional reduction of a minkowskian theory in dimension $(1, d)$ over time, $(1, d) \to (0, d)$.[6] In general, dimensional reduction modifies the geometry of the scalar manifold $\mathcal{M}$, if (i) components of tensor fields or gauge fields become scalars or if (ii) one obtains a field strength of rank $d - 1$. In the latter case one can Hodge-dualize the field strength into a field strength of rank 1, or, in other words, the derivative of a scalar field. Both phenomena are well known for dimensional reduction over space, and dimensional reduction over time adds an interesting twist.

Let us give a simple example for case (i), which is the mechanism relevant for the main part of this paper. For definiteness, we start with one real scalar field σ and one abelian gauge field A_μ in dimension $(1, 4)$. Then the dimensional reduction of the lagrangian[7]

$$\mathcal{L}^{(1,4)} = -\frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{1.1}$$

over space gives

$$\mathcal{L}^{(1,3)} = -\frac{1}{2}\partial_\mu \sigma \partial^\mu \sigma - \frac{1}{2}\partial_\mu b \partial^\mu b - \frac{1}{4} F_{mn} F^{mn} \,. \tag{1.2}$$

Here F_{mn} is the field strength of the $(1, 3)$-dimensional abelian gauge field A_m obtained by decomposing $A_\mu = (A_m, A_5)$. The component A_5 gives rise to the additional scalar field b. Defining $z = \sigma + ib$, this can be rewritten as

$$\mathcal{L} = -\frac{1}{2}\partial_\mu z \partial^\mu \overline{z} - \frac{1}{4} F_{mn} F^{mn} \,. \tag{1.3}$$

This shows that the complex coordinate z defines a *complex structure* on the scalar manifold, with respect to which the target metric defined by the scalar couplings is *kählerian*.

The same mechanism carries over to interacting theories, in particular to those involving non-linear sigma models, where the scalar fields can be interpreted as maps from space-time into a riemannian manifold $\mathcal{M}$. If the theory is supersymmetric, the geometry of $\mathcal{M}$ is subject to restrictions, the details of which depend on the underlying supermultiplet. The minimal supersymmetric extension of (1.1) is a theory of abelian vector multiplets in

[6] We say that a space-time has dimension (t, s), if it has t time-like and s space-like dimensions. We will consider the case $d = 4$, but the remarks in this paragraph apply to general d.

[7] We use the 'mostly plus' convention where the metric is negative definite in the time-like directions. With this convention standard kinetic terms for tensor gauge fields always have a minus sign in front.

dimension $(1,4)$, which reduces to an $\mathcal{N}=2$ supersymmetric theory of vector multiplets in dimension $(1,3)$. The allowed target manifolds $\mathcal{M}$ for the latter theories are the *affine special Kähler manifolds* [34]–[39].[8]

Dimensional reduction of (1.1) over time gives instead

$$\mathcal{L}^{(0,4)} = -\frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \frac{1}{2}\partial_\mu b\partial^\mu b - \frac{1}{4}F_{mn}F^{mn}\,. \tag{1.4}$$

Introducing an object e with the properties $e^2=1$ and $\overline{e}=-e$ [5], and defining $z=\sigma+eb$, we see that (1.4) takes the same form as (1.3), but with a different interpretation of the 'complex' scalar field z. As we will explain in detail in the main part of the paper, the scalar target space of the euclidean $(0,4)$-dimensional theory carries a *para-complex structure*, for which z is a *para-holomorphic coordinate.*

The main objective of this paper is to generalize the above simple example to supersymmetric models with a curved scalar manifold. In order to characterize the allowed target manifolds, we introduce in section 2 the notion of a (rigid or affine) *special para-Kähler manifold* and investigate systematically the corresponding geometry. Our fundamental result about such manifolds is that any simply connected affine special para-Kähler manifold M admits a para-kählerian lagrangian immersion into a symplectic para-complex vector space V endowed with a compatible para-complex conjugation. The immersion induces the special geometric structures on M and is unique up to an affine transformation of V which preserves the symplectic structure and the para-complex conjugation. As a corollary, we obtain that any affine special para-Kähler manifold M of para-complex dimension n is locally described by a para-holomorphic *prepotential* F of n para-complex variables, the Hessian of which has to satisfy a certain non-degeneracy condition. In contrast with the para-kählerian lagrangian immersion, which is unique up to an affine transformation, the prepotential depends nonlinearly on the choice of a compatible lagrangian splitting of V (choice of coordinates and momenta in the symplectic para-complex vector space V).

In section 3 we discuss various properties of fermions and of supersymmetry algebras in dimensions $(1,4),(1,3)$ and $(0,4)$, which we need for the dimensional reduction. We consider this both from a physicist's and from a mathematician's perspective by combining [40] with [41, 42]. Particular attention is payed to symplectic Majorana spinors and to the R-symmetry groups of the relevant supersymmetry algebras. These play a crucial role, since symplectic Majorana spinors allow us to write the fermionic terms in dimensions $(1,4),(1,3)$ and $(0,4)$ in a uniform way, while R-symmetry is related to the existence of a (para-)complex structure on the scalar manifold.

In section 4 we construct the general lagrangian of rigid vector multiplets in dimension (1,4) by adapting the results obtained in [43] for superconformal vector multiplets. The couplings are encoded in a real (not necessarily homogeneous) cubic prepotential, and the metric of the scalar manifold is the Hessian of this function, as was also found in [44]. This provides the definition of an *affine very special real manifold*, which is the affine version of the very special real manifolds considered in [45, 46, 47].

[8]These manifolds are also called rigid special Kähler manifolds.

In section 5 we perform the dimensional reduction of the $(1,4)$-dimensional lagrangian over space and time in parallel. While the reduction over space gives an affine special Kähler manifold, the reduction over time results in an affine special para-Kähler manifold. The mapping of scalar manifolds induced by the dimensional reduction $(1,4) \to (1,3)$ is known as the r-map [46]. Here we encounter a new mapping induced by dimensional reduction over time $(1,4) \to (0,4)$, which we call *the temporal r-map*. It associates an affine special para-Kähler manifold to any affine very special real manifold. To get the general 4-dimensional lagrangian we proceed in two steps. First we observe that although the explicit lagrangian and supersymmetry rules obtained by dimensional reduction depend on a cubic prepotential, all these formulae have a well defined geometrical meaning for general (para-)holomorphic prepotentials. In particular, the prepotential is the generating function of a (para-)holomorphic lagrangian immersion which induces the special geometric structures on the scalar manifold. For euclidean space-time signature this follows from the results of section 2. Therefore we can consider 4-dimensional lagrangians defined by a a general (para-)holomorphic prepotential. However, these lagrangians are not supersymmetric anymore, because supersymmetry variations now generate terms which contain the fourth derivative of the prepotential. Therefore the second step consists in adding further terms to the lagrangian, which restore its invariance under supersymmetry. It turns out that it is sufficient (in the off-shell formulation) to add one particular four-fermion term, which contains the fourth derivative of the prepotential. For the case of Minkowski space-time signature the general $\mathcal{N} = 2$ vector multiplet lagrangian is of course well known [26, 27, 36]. We check our result by comparing it to [26, 27, 48] and find complete agreement. The advantage of our formalism is that the minkowskian and euclidean lagrangian and supersymmetry transformations take the same form, when written in holomorphic and para-holomorphic coordinates, respectively. Therefore it follows immediately that every para-holomorphic prepotential defines a supersymmetric euclidean lagrangian, irrespective of whether this theory can be obtained by dimensional reduction or not. Moreover, we see that we have found the general euclidean vector multiplet lagrangian.

Given the mathematical results of section 2, we could have approached this result more directly, without invoking dimensional reduction. All what is needed in addition to section 2 is the geometric interpretation of the fermions and gauge fields as bundle-valued sections of the para-complexified tangent bundle of the target manifold. We preferred to proceed through dimensional reduction for two reasons, which already mentioned above: first, in various applications we are interested in knowing explicitly how theories are related by dimensional reduction or oxidation. Second, dimensional reduction generates a dictionary between the minkowskian and euclidean theory. The geometrical structures can be read off easily and it is obvious how to bring the lagrangians of both space-time signatures to the same form.

The result that every para-holomorphic prepotential defines a euclidean supersymmetric lagrangian does not imply that every such theory can be obtained from a minkowskian theory by analytic continuation. The reason is that not every para-holomorphic function can be obtained by analytic continuation of a holomorphic function. This is not even possible for para-holomorphic functions which are real-valued on real points. In fact, a

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holomorphic function is automatically real-analytic, while a para-holomorphic function need not be real-analytic.[9] The euclidean theories which can be obtained by analytic continuation of minkowskian theories therefore form a subset. They are defined by a para-holomorphic prepotential which is real-valued and real-analytic on real points. There are two obvious ways to find explicit examples of such pairs of theories. The first approach is dimensional reduction. We have already seen that theories with a cubic prepotential come in pairs. This procedure can be generalized by considering the full 5-dimensional theory in the background $M_4 \times S^1$, where S^1 is a space-like or time-like circle of finite radius. In this case the massive Kaluza-Klein modes do not decouple completely and induce effective interactions in the action of the massless modes. Gauge theories and string theories on $M_4 \times S^1$, with space-like S^1, have been studied in the literature, see for example [50, 49, 51, 52]. The second approach is more general and intrinsically 4-dimensional. One can adapt and generalize the continuous Wick rotation of [22, 23] to $\mathcal{N} = 2$ theories with suitable non-trivial scalar manifolds. We will discuss this approach, and its relation to dimensional reduction, in a future publication.

Finally, in section 5.4, we show that the occurrence of a para-complex structure in euclidean space-time signature is a consequence of the non-compactness of the abelian factor of the euclidean R-symmetry group. We also see that the general vector multiplet lagrangian in either signature is only invariant under a discrete $\mathbb{Z}_2$-subgroup of this factor. This is as expected from the fact that the continuous abelian factor of the R-symmetry group is anomalous, and therefore generically broken by quantum effects.

We have organized the material such that the focus is on the mathematical aspects in sections 2 and 3 and on the physical aspects in sections 4 and 5. The sections are as self-contained as possible. Readers who want to approach the physical aspects directly can continue reading at section 4 and then go back to sections 2 and 3 for the underlying mathematics.

1.3 Outlook

As we have seen above, the geometrical structure underlying the $i \to e$ substitution rule is the *replacement of a complex by a para-complex structure.* In this paper the case of rigid vector multiplets is treated in detail, and it is already clear how this result is to be extended in various directions. In the following we will indicate the most immediate extensions in the context of theories with $\mathcal{N} = 2$ supersymmetry (8 supercharges). But before doing so, let us note that para-complex structures also appear in other situations, in particular in the first example for the $i \to e$ substitution rule, 10-dimensional IIB supergravity [5]. Here the complex manifold $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is replaced by the para-complex manifold $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(1,1)$.[10]

The next step in our program will be the further reduction to 3 dimensions. For dimensional reduction over space, $(1,3) \to (1,2)$, it is well known that the resulting lagrangian can be expressed in terms of hypermultiplets [46, 53]. This uses that one can dualize a

[9] We assume throughout that the prepotential is C^∞.

[10] In fact, these symmetric manifolds are projective special Kähler and projective special para-Kähler, respectively.

3-dimensional gauge field into a scalar, which is a particular case of the mechanism (ii) mentioned above. Supersymmetry requires that the scalar geometry of hypermultiplets in rigid supersymmetry must be hyper-Kähler [54]. Dimensional reduction induces a map which assigns to each affine special Kähler manifold a hyper-Kähler manifold. This is known as the c-map [46, 53]. Dimensional reduction over time yields a version of it, which we call the temporal c-map. As we will explain in detail in a future publication, dimensional reduction over time has the effect that one obtains one complex and two para-complex structures, leading to *para-hyper-Kähler manifolds.*

The obvious next step is then to consider vector and hypermultiplets coupled to supergravity. The geometry of locally supersymmetric vector multiplets in dimension (1,3) is known as projective special Kähler geometry [55, 56, 57, 36, 37, 38, 39].[11] As indicated by the name, such manifolds can be obtained from affine special Kähler manifolds (with suitable homogeneity properties) by projectivization. This can also be understood from the physical point of view in terms of the conformal calculus. Here one first constructs a superconformally invariant theory and then eliminates the so-called conformal compensators by imposing gauge conditions. This gauge fixing amounts to the projectivization of the scalar manifold underlying the superconformal theory. Based on the results obtained in this paper, one can adapt this construction to the case of euclidean signature and construct *projective special para-Kähler manifolds.* Similar remarks apply to hypermultiplets coupled to supergravity. In Minkowski signature the coupling to supergravity implies that the scalar geometry is quaternionic-Kähler instead of hyper-Kähler [58]. The relation between these two kinds of geometries can again be understood as projectivization, because every quaternionic-Kähler manifold can be obtained as the quotient of a hyper-Kähler cone [59]. This relation is natural from the viewpoint of conformal calculus [60] and it is possible to adapt the construction to theories with euclidean signature in order to construct *para-quaternionic-Kähler manifolds.* In fact, such manifolds have already occurred in the context of instantons in IIB string theory compactified on a Calabi-Yau threefold. Here it was found that in euclidean signature the Hodge-dualization of the universal double-tensor multiplet gives a hypermultiplet with scalar manifold is $\mathrm{SL}(3,\mathbb{R})/(\mathrm{SL}(2,\mathbb{R})\otimes \mathrm{SO}(1,1))$ [7]. This is a symmetric para-quaternionic-Kähler manifold.

2. Para-complex geometry

2.1 Para-complex manifolds

Definition 1. *Let V be a finite dimensional (real) vector space. A* para-complex structure *on V is a nontrivial involution $I \in \mathrm{End}\, V$, i.e., $I^2 = \mathrm{Id}$ and $I \neq \mathrm{Id}$, such that the two eigenspaces $V^{\pm} := \ker(\mathrm{Id} \mp I)$ of I are of the same dimension. A* para-complex vector space *is a vector space endowed with a para-complex structure. A* homomorphism *from a para-complex vector space (V,I) into a para-complex vector space (V',I') is a linear map $\phi : (V,I) \to (V',I')$ satisfying $\phi I = I'\phi$.*

[11] This is also called local special Kähler geometry in the physics literature.

Let I be a para-complex structure on a vector space V. Then there exists a basis $(e_1^+, \ldots, e_n^+, e_1^-, \ldots, e_n^-)$ of V, $\dim V = 2n$, such that $Ie_i^\pm = \pm e_i^\pm$ and we can identify I with the diagonal matrix

$$I = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1) = \begin{pmatrix} \mathbf{1}_n & 0 \\ 0 & -\mathbf{1}_n \end{pmatrix}.$$

It follows that the automorphism group $\mathrm{Aut}(V, I) := \{L \in \mathrm{GL}(V) | \, LIL^{-1} = I\}$ of (V, I), which coincides with the centralizer $Z_{\mathrm{GL}(V)}(I)$ of I in $\mathrm{GL}(V)$, is given by

$$\mathrm{Aut}(V, I) = Z_{\mathrm{GL}(V)}(I) := \{L \in \mathrm{GL}(V) | \, [L, I] = 0\} = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \middle| \, A, D \in \mathrm{GL}_n(\mathbb{R}) \right\}.$$

Definition 2. *An* almost para-complex structure *on a smooth manifold M is an endomorphism field $I \in \Gamma(\mathrm{End}\, TM)$, $p \mapsto I_p$, such that I_p is a para-complex structure on T_pM for all $p \in M$. In other words:*

(i) $I \neq \mathrm{Id}_{TM}$ is an involution, i.e. $I^2 = \mathrm{Id}_{TM}$ and

(ii) the two eigendistributions $T^\pm M := \ker(\mathrm{Id} \mp I)$ of I have the same rank.

An almost para-complex structure I is called integrable *if the distributions $T^\pm M$ are both integrable. An integrable almost para-complex structure is called a* para-complex structure. *A manifold M endowed with a para-complex structure is called a* para-complex manifold. *The* Nijenhuis tensor *N_I of an almost para-complex structure I is the tensor defined by the equation:*

$$N_I(X, Y) := [X, Y] + [IX, IY] - I[X, IY] - I[IX, Y],$$

for all vector fields X and Y on M.

Proposition 1. *An almost para-complex structure I is integrable if and only if $N_I = 0$.*

Proof. Consider the two projections $\pi_\pm : TM \to T^\pm M$, $\pi_\pm := \frac{1}{2}(\mathrm{Id} \pm I)$. Then, by the Frobenius theorem, the integrability of T^+M and T^-M is equivalent to, respectively,

$$\pi_-[\pi_+X, \pi_+Y] = 0 \qquad \text{and} \qquad \pi_+[\pi_-X, \pi_-Y] = 0,$$

for all vector fields X and Y. The sum and the difference of these expressions are proportional to $N_I(X, Y)$ and $IN_I(X, Y)$ respectively. □

Example 1: any para-complex vector space (V, I) can be considered as a para-complex manifold, with constant para-complex structure.

Example 2: the cartesian product $M \times N$ of two para-complex manifolds (M, I_M) and (N, I_N) is a para-complex manifold with the para-complex structure $I_{M\times N} := I_M \oplus I_N$. Here we have used the identification $T(M \times N) = TM \oplus TN$.

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Example 3: let $M = M_+ \times M_-$ be the cartesian product of two smooth manifolds M_+ and M_- of the same dimension. We can identify $T_{(p_+,p_-)}M = T_{p_+}M_+ \oplus T_{p_-}M_-$ and define a para-complex structure I on M by $I|T_{p_\pm}M_\pm := \pm\mathrm{Id}$. The next result shows that any para-complex manifold is locally of this form.

Proposition 2. *Let (M, I) be a para-complex manifold of dimension $2n$. Then for every point in M there is an open neighborhood U and a diffeomorphism $\phi : U \cong M_+ \times M_-$ onto the product of two manifolds $M_\pm \cong \mathbb{R}^n$ such that ϕ maps the leaves of the foliation T^+M to the submanifolds $M_+ \times \{p_-\}$, $p_- \in M_-$, and the leaves of T^-M to the submanifolds $\{p_+\} \times M_-$, $p_+ \in M_+$.*

Proof. By the Frobenius theorem applied to the distribution T^-M, there exists an open neighborhood U of p and functions z_+^i, $i = 1, \ldots, n$, on U, which are constant on the leaves of T^-M and such that the differentials dz_+^i are (point-wise) linearly independent. Similarly, by restricting U, if necessary, we can find functions z_-^i, $i = 1, \ldots, n$, on U constant on the leaves of T^+M and such that the differentials dz_-^i are linearly independent. From the transversality of the two foliations T^+M and T^-M we conclude that $(z_+^1, \ldots, z_+^n, z_-^1, \ldots, z_-^n)$ is a system of local coordinates. Now we can define $\phi := (\phi_+, \phi_-) : U \to \mathbb{R}^n \times \mathbb{R}^n$ by $\phi_\pm = (z_\pm^1, \ldots, z_\pm^n)$. By restricting U, if necessary, we can assume that ϕ is a diffeomorphism onto a product $M_+ \times M_-$ of two open sets $M_\pm \cong \mathbb{R}^n$ in $\mathbb{R}^n$. $\square$

Coordinates $(z_+^1, \ldots, z_+^n, z_-^1, \ldots, z_-^n)$ as in the proof of proposition 2 will be called *adapted* to the para-complex structure. We put

$$x^i := \frac{z_+^i + z_-^i}{2} \qquad \text{and} \qquad y^i := \frac{z_+^i - z_-^i}{2}\,.$$

We wish to think of x^i and y^i as 'real' and 'imaginary' parts of a system of local 'para-holomorphic' coordinates z^i, $i = 1, \ldots, n$.

In order to make this analogy precise, we introduce the algebra C of *para-complex numbers*. It is the (real) algebra generated by 1 and the symbol e subject to the relation $e^2 = 1$. If we think of e as a unit vector in a one-dimensional vector space with a negative definite scalar product, then C is just the corresponding Clifford algebra $C = C\ell_{0,1} \cong \mathbb{R} \oplus \mathbb{R}$, the same as $C\ell_{1,0} \cong \mathbb{C}$ gives the field of complex numbers. The map $\bar{\cdot} : C \to C$, $x + ey \mapsto x - ey$, where $x, y \in \mathbb{R}$, is called the *para-complex conjugation.* It is a C-antilinear involution: $\overline{ez} = -e\bar{z}$. The real numbers $x = \operatorname{Re} z := (z + \bar{z})/2$ and $y = \operatorname{Im} z := e(z - \bar{z})/2$ are called *real* and *imaginary* parts of $z = x + ey$, respectively. Note that $z\bar{z} = x^2 - y^2$. Therefore C is sometimes called the algebra of *hyperbolic* complex numbers. Note also that the group $C^* \subset C$ of invertible elements has 4 connected components. In fact, any $z \in C^*$ can be written as $z = \pm r \exp(et)$ or as $z = \pm er \exp(et)$, where $r > 0$ and $t \in \mathbb{R}$. This is the analogue of the polar decomposition $z = r \exp(it)$ for $z \in \mathbb{C}^* \cong \mathbb{R}^{>0} \times S^1$. The circle $S^1 = \{z = x + iy \in \mathbb{C} | x^2 + y^2 = 1\}$ is replaced by the four hyperbolas $\{z = x + ey \in C | x^2 - y^2 = \pm 1\}$.

A *C-valued smooth function* on a smooth manifold is a smooth map $f : M \to C \cong \mathbb{R}^2$. For such functions we can define the C-valued smooth function $\overline{f}$ and the real-valued functions $\operatorname{Re} f$ and $\operatorname{Im} f$ on M. Obviously, C is a para-complex vector space. Its para-complex structure is simply multiplication by e. Moreover, the eigenspaces $C^{\pm} = \mathbb{R}(1 \pm e) \cong \mathbb{R}$ of e are precisely the two simple ideals of the algebra $C = C^+ \times C^-$. More generally, the free C-module C^n is a para-complex vector space. Its para-complex structure is the scalar multiplication by $e \in C$.

Definition 3. *A smooth map $\phi : (M, I_M) \to (N, I_N)$ between para-complex manifolds is called* para-holomorphic *(or just holomorphic, when no confusion is possible), if $d\phi I_M = I_N d\phi$. A para-holomorphic map $f : (M, I) \to C$ is called a* para-holomorphic function. *The* para-complex dimension *of a para-complex manifold M is the integer* $\dim_C M := \dim M/2$. *A system of para-holomorphic functions z^i, $i = 1, \ldots, n$, defined on some open subset $U \subset M$ of a para-complex manifold is called a* system of local para-holomorphic coordinates *(or just system of local holomorphic coordinates) if $(x^1 = \operatorname{Re} z^1, \ldots, x^n = \operatorname{Re} z^n, y^1 = \operatorname{Im} z^1, , \ldots, y^n = \operatorname{Im} z^n)$ is a system of (real) local coordinates.*

Notice that a collection of para-holomorphic functions $z^1, \ldots, z^n$, defined in a neighborhood of a point $p \in M$, forms a system of local coordinates in some neighborhood of p if and only the differentials $dz^i_p, d\overline{z}^i_p$, $i = 1, \ldots, n$, form a basis of the free C-module $T^*_p M \otimes C$. This is an easy consequence of the fact that $dz^i I = e dz^i$, i.e.

$$dx^i I = dy^i \qquad \text{and} \qquad dy^i I = dx^i \, .$$

A system of local para-holomorphic coordinates defined over $U \subset M$ defines an isomorphism from the para-complex manifold U (with the para-complex structure induced by the inclusion) onto some open subset of the para-complex vector space C^n.

Proposition 3. *Let M be a smooth manifold endowed with an atlas $\mathcal{A} = \{\phi : U \to C^n = \mathbb{R}^{2n}\}$ such that the coordinate changes are para-holomorphic. Then there exists a unique para-complex structure $I_{\mathcal{A}}$ on M such that all $\phi \in \mathcal{A}$ are para-holomorphic. Conversely, any para-complex manifold admits such an atlas.*

Proof. Given a smooth manifold with an atlas as above, we can pull back the para-complex structure from C^n to M. This gives a well defined para-complex structure on M, since the coordinate changes are para-holomorphic. Conversely, let (M, I) be a para-complex manifold. Any adapted system of coordinates $(z^1_+, \ldots, z^n_+, z^1_-, \ldots, z^n_-) : U \to \mathbb{R}^{2n}$ defines a system of para-holomorphic coordinates $(z^1, \ldots, z^n)$ by $\operatorname{Re} z^i := x^i = (z^i_+ + z^i_-)/2$ and $\operatorname{Im} z^i := y^i = (z^i_+ - z^i_-)/2$. In fact, from the definition of $z^i_\pm$ it follows easily that $dz^i_\pm \circ I = \pm dz^i_\pm$. Using the above equations, this implies that $dx^i \circ I = dy^i$, and hence $dz^i \circ I = (dx^i + e dy^i) \circ I = dy^i + e dx^i = e dz^i$. This shows that the functions z^i are indeed para-holomorphic. It is clear that their real and imaginary parts form a (real) coordinate system, since the $z^i_\pm$ do. Now it suffices to observe that we can cover M by coordinate domains U as above and that the coordinate changes are para-holomorphic. $\square$

Notice that a C-valued function $f : M \to C$ on a para-complex manifold (M, I) is para-holomorphic if and only if it satisfies the partial differential equations

$$\frac{\partial f}{\partial \bar{z}^i} := \frac{1}{2}\left(\frac{\partial f}{\partial x^i} - e\frac{\partial f}{\partial y^i}\right) = 0\,,$$

where the (z^i) are local para-holomorphic coordinates. The general solution of this system is of the form

$$\mathrm{Re}\, f = f_+ + f_-\,, \qquad \mathrm{Im}\, f = f_+ - f_-\,, \qquad \frac{\partial f_\pm}{\partial z^i_\mp} = 0\,.$$

In other words f_+ is a function which, in adapted coordinates $(z^i_\pm)$, depends only on the $z^i_+ = x^i + y^i$ and f_- depends only the $z^i_- = x^i - y^i$.

Para-holomorphic vector bundles

Definition 4. *A* para-holomorphic vector bundle *of rank r is a (smooth) real vector bundle $\pi : W \to M$ of rank $2r$ whose total space W and base M are para-complex manifolds and whose projection π is a para-holomorphic map.*

From the fact that the projection π is para-holomorphic it follows that the para-complex structure on W induces on each fibre $W_p = \pi^{-1}(p)$, $p \in M$, the structure of a para-complex vector space of para-complex dimension r. In particular, we have a canonical splitting $W = W^+ \oplus W^-$, where $W_p^\pm$ are the two eigenspaces of the para-complex structure on $W_p^\pm$, for all $p \in M$.

Example 4: the tangent bundle $TM \to M$ over any para-complex manifold M is a para-holomorphic vector bundle and we have $(TM)^\pm = T^\pm M$.

Example 5: sums and duals of para-holomorphic vector bundles are again para-holomorphic vector bundles. Also tensor products, over the commutative algebra C, of paraholomorphic vector bundles over the same base M are again para-holomorphic vector bundles over M.

The decomposition $T^*M = (T^*M)^+ \oplus (T^*M)^-$, defined for any almost para-complex manifold, induces a bigrading on exterior forms:

$$\wedge^k T^*M = \oplus_{p+q=k} \wedge^{p+,q-} T^*M\,.$$

In particular, $\wedge^{1+,0-} T^*M = (T^*M)^+$ and $\wedge^{0+,1-} T^*M = (T^*M)^-$. We will say that a k-form ω is of *type* $(p+, q-)$ if $\omega \in \wedge^{p+,q-} T^*M$. The corresponding decomposition of differential forms on M will be denoted by

$$\Omega^k(M) = \oplus_{p+q=k} \Omega^{p+,q-}(M)\,.$$

If the almost para-complex structure is integrable, then the de Rham differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ splits as $d = \partial_+ + \partial_-$, where

$$\partial_+ : \Omega^{p+,q-}(M) \to \Omega^{(p+1)+,q-}(M) \qquad \text{and} \qquad \partial_- : \Omega^{p+,q-}(M) \to \Omega^{p+,(q+1)-}(M)\,.$$

Moreover, we have $\partial_+^2 = \partial_-^2 = \partial_+\partial_- + \partial_-\partial_+ = 0$. As a consequence, the differential operator ∂_+ (or ∂_-) can be used to define a real version of Dolbeault cohomology on any para-complex manifold.

Consider now the *para-complexification* $TM^C := TM \otimes C$ of the tangent bundle TM of an almost para-complex manifold (M, I). We extend the almost para-complex structure $I : TM \to TM$ to a C-linear endomorphism field $I : TM^C \to TM^C$. Then for all $p \in M$ the free C-module T_pM^C has the following canonical decomposition into the direct sum of two free C-modules

$$T_pM^C = T_p^{1,0}M \oplus T_p^{0,1}M\,,$$

where

$$T_p^{1,0}M := \{X + eIX | X \in T_pM\} \qquad \text{and} \qquad T_p^{0,1}M := \{X - eIX | X \in T_pM\}\,.$$

The subbundles $T^{1,0}M$ and $T^{0,1}M$ are characterized as the $\pm e$-eigenbundles of $I : TM^C \to TM^C$, in the sense that $I = e$ on $T^{1,0}M$ and $I = -e$ on $T^{0,1}M$. The canonical isomorphism $TM \to T^{1,0}M$, $X \to X^{1,0} := \frac{1}{2}(X + eIX)$, of real vector bundles is compatible with the para-complex structures on the fibres; $(IX)^{1,0} = eX^{1,0}$. In particular, it is an isomorphism of para-holomorphic vector bundles if the almost para-complex structure I happens to be integrable. Note that the isomorphism $TM \to T^{0,1}M$, $X \to X^{0,1} := \frac{1}{2}(X - eIX)$, of real vector bundles maps I to $-e$; $(IX)^{0,1} = -eX^{0,1}$. Consider now the $\pm e$-eigenbundles of $I^* : T^*M^C \to T^*M^C$:

$$\wedge^{1,0}T^*M := \{\alpha + eI^*\alpha | \alpha \in T^*M\} \qquad \text{and} \qquad \wedge^{0,1}\,T^*M := \{\alpha - eI^*\alpha | \alpha \in T^*M\}\,.$$

The decomposition $T^*M^C := \wedge^{1,0}T^*M \oplus \wedge^{0,1}T^*M$ induces a bigrading on C-valued exterior forms:

$$\wedge^k T^*M^C = \oplus_{p+q=k} \wedge^{p,q}\,T^*M\,.$$

We will say that a C-valued k-form ω is of *type* (p, q) if $\omega \in \wedge^{p,q}T^*M$. The corresponding decomposition of C-valued differential forms on M will be denoted by

$$\Omega^k_C(M) = \oplus_{p+q=k}\Omega^{p,q}(M)\,.$$

Notice that the real vector bundles $\wedge^{p,0}T^*M$ are para-holomorphic with the para-complex structure on the fibres defined by e, if the almost para-complex structure I is integrable. Suppose now that I is integrable. Then the (C-linearly extended) de Rham differential $d : \Omega^k_C(M) \to \Omega^{k+1}_C(M)$ splits as $d = \partial + \overline{\partial}$, where

$$\partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M) \qquad \text{and} \qquad \overline{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\,.$$

Moreover, $\partial^2 = \overline{\partial}^2 = \partial\overline{\partial} + \overline{\partial}\partial = 0$. As a consequence, the differential operator $\overline{\partial}$ can be used to define a para-complex version of Dolbeault cohomology.

2.2 Para-Kähler manifolds

Definition 5. *Let (V, I) be a para-complex vector space. A pseudo-euclidean scalar product g on V is called* para-hermitean *if I is an anti-isometry for g, i.e.,*

$$I^*g = g(I\cdot, I\cdot) = -g\,.$$

A para-hermitean vector space *is a para-complex vector space endowed with a para-hermitean scalar product. The pair (I, g) is called a* para-hermitean structure *on the vector space V.*

Example 6: consider the vector space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ with its standard basis $(e_i^{\pm})$, where $e_i^+ := e_i \oplus 0$ and $e_i^- = 0 \oplus e_i$, and its standard para-complex structure, given by $Ie_i^{\pm} = \pm e_i^{\pm}$. We can define a para-hermitean scalar product g by $g(e_i^{\pm}, e_j^{\pm}) = 0$ and $g(e_i^{\pm}, e_j^{\mp}) = \delta_{ij}$. We will call (I, g) the *standard para-hermitean structure* of $\mathbb{R}^{2n}$.

Example 7: consider the vector space $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ with its standard basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$, where $f_i = ee_i$, and its standard para-complex structure, given by $Ie_i = f_i$ and $If_i = e_i$. We can define a para-hermitean scalar product g by $g(e_i, e_j) = -g(f_i, f_j) = \delta_{ij}$ and $g(e_i, f_j) = 0$. We will call (I, g) the *standard para-hermitean structure* of C^n.

Any two para-hermitean vector spaces (V, I, g) and (V', I', g') of the same dimension are isomorphic, i.e., there exists a linear para-holomorphic isometry $\phi : V \to V'$. In particular, $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ and C^n are isomorphic as para-hermitean vector spaces. An isomorphism $\phi : R^{2n} \to C^n$ is given by

$$\phi e_i^{\pm} = \frac{1}{\sqrt{2}}(e_i \pm f_i)\,.$$

The model $\mathbb{R}^{2n}$ has the advantage that I is diagonal in the standard basis, whereas g is diagonal in the standard real basis of C^n.

Definition 6. *Let V be a para-hermitean vector space with para-hermitean structure (I, g). The* para-unitary group *of V is the automorphism group*

$$\mathrm{U}^{\pi}(V) := \mathrm{Aut}(V, I, g) = \{L \in \mathrm{GL}(V) \,|\, [L, I] = 0 \qquad \textit{and} \qquad L^* g = g\}\,.$$

Proposition 4. *(i) The para-unitary group of the para-hermitean vector space $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ is given by*

$$\mathrm{U}^{\pi}(\mathbb{R}^{2n}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix} \middle| \, A \in \mathrm{GL}_n(\mathbb{R}) \right\} \cong \mathrm{GL}_n(\mathbb{R})\,,$$

where $A^{-t} := (A^{-1})^t = (A^t)^{-1}$ is the contragredient matrix.

(ii) The para-unitary group $\mathrm{U}^{\pi}(C^n) \cong \mathrm{U}^{\pi}(\mathbb{R}^{2n}) \cong \mathrm{GL}_n(\mathbb{R})$ of the para-hermitean vector space $C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$ is given by

$$\mathrm{U}^{\pi}(C^n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \middle| \, A, B \in \mathrm{End}(\mathbb{R}^n)\,,\; A^t A - B^t B = \mathbf{1}_n\,,\; A^t B - B^t A = 0 \right\}.$$

Proof. (i) The centralizer of I consists of block diagonal matrices $L = \mathrm{diag}(A, D)$, $A, D \in \mathrm{GL}_n(\mathbb{R})$. It is sufficient to check that $L^* g = g$ is equivalent to $A^t D = 1_n$.

(ii) The centralizer of I consists of matrices of the form

$$L = \begin{pmatrix} A & B \\ B & A \end{pmatrix}.$$

The equation $L^*g = g$ is now equivalent to $A^tA - B^tB = \mathbf{1}_n$ and $A^tB - B^tA = 0$. Since the para-hermitean vector spaces $\mathbb{R}^{2n}$ and C^n are isomorphic, the corresponding para-unitary groups are isomorphic as well. An explicit isomorphism $\mathrm{U}^\pi(\mathbb{R}^{2n}) \xrightarrow{\sim} \mathrm{U}^\pi(C^n)$ is

$$\mathrm{diag}(A, D) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \to \begin{pmatrix} \frac{A+D}{2} & \frac{A-D}{2} \\ \frac{A-D}{2} & \frac{A+D}{2} \end{pmatrix},$$

where $D = A^{-t}$. □

Notice that the para-complex structure I does not belong to the para-unitary group, simply because $I^*g = -g \neq g$. However, it belongs to the *para-unitary Lie algebra* $\mathfrak{u}^\pi(V) = Lie\, \mathrm{U}^\pi(V)$ and generates a closed non-compact central subgroup $\{\exp tI | t \in \mathbb{R}\} \cong \mathbb{R}^{>0}$ of the para-unitary group. If V is a para-complex vector space of para-complex dimension 1, i.e., $V \cong C \cong \mathbb{R}^2$, then $\mathrm{U}^\pi(V) = \{\pm \exp tI | t \in \mathbb{R}\} \cong \mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^*$.

Definition 7. *An* almost para-hermitean manifold (M, I, g) *is an almost para-complex manifold* (M, I) *endowed with a pseudo-riemannian metric* g *such that* $I^*g = -g$*. If* I *is integrable, we say that* (M, I, g) *is a* para-hermitean manifold*. The two-form* $\omega := g(I\cdot, \cdot) = -g(\cdot, I\cdot)$ *is called the* fundamental two-form *of the almost para-hermitean manifold* (M, I, g)*.*

Notice that the fundamental two-form satisfies $I^*\omega = -\omega$, and is hence of type $(1+, 1-)$ or, equivalently, of type $(1, 1)$ (when considered as C-valued two-form).

Definition 8. *A* para-Kähler manifold (M, I, g) *is an almost para-hermitean manifold* (M, I, g) *such that* I *is parallel with respect to the Levi-Civita connection* D *of* g*, i.e.,* $DI = 0$*.*

The condition $DI = 0$ easily implies $N_I = 0$ and $d\omega = 0$. Therefore any para-Kähler manifold (M, I, g) is a para-hermitean manifold with closed fundamental form ω. The symplectic form ω will be called the *para-Kähler form*.

The next theorem characterizes para-Kähler manifolds as para-hermitean manifolds with closed fundamental form.

Theorem 1. *Let* (M, I, g) *be a para-hermitean manifold with closed fundamental form* ω*. Then* (M, I, g) *is a para-Kähler manifold. Conversely, any para-Kähler manifold is a para-hermitean manifold with closed fundamental form.*

Proof. Let (M, I, g) be a para-hermitean manifold with closed fundamental form ω and D the Levi-Civita connection of g. It is determined by the Koszul formula

$$2g(D_XY, Z) = Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) + g([X, Y], Z) - g([X, Z], Y) - g([Y, Z], X),$$

which holds for all vector fields X, Y and Z on M. We have to show that $DI = 0$. Therefore we compute

$$2g((D_XI)Y, Z) = 2g(D_X(IY), Z) - 2g(ID_XY, Z)$$

$$
\begin{aligned}
&= 2g(D_X(IY),Z) + 2g(D_X Y, IZ)\\
&= Xg(IY,Z) + IYg(X,Z) - Zg(X,IY)\\
&\quad +g([X,IY],Z) - g([X,Z],IY) - g([IY,Z],X) +\\
&\quad +Xg(Y,IZ) + Yg(X,IZ) - IZg(X,Y)\\
&\quad +g([X,Y],IZ) - g([X,IZ],Y) - g([Y,IZ],X)\\
&= (X\omega(Y,Z) + Y\omega(Z,X) + Z\omega(X,Y) -\\
&\quad -\omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y)) + \omega([Y,Z],X) +\\
&\quad +IYg(X,Z) + Xg(Y,IZ) - IZg(X,Y) +\\
&\quad +g([X,IY],Z) - g([IY,Z],X) - g([X,IZ],Y) - g([Y,IZ],X)\\
&= d\omega(X,Y,Z) + \omega([Y,Z],X) + (X\omega(IY,IZ) + IY\omega(IZ,X) +\\
&\quad +IZ\omega(X,IY) - \omega([X,IY],IZ) - \omega([IY,IZ],X) - \omega([IZ,X],IY)) +\\
&\quad +\omega([IY,IZ],X) - g([IY,Z],X) - g([Y,IZ],X)\\
&= d\omega(X,Y,Z) + \omega([Y,Z],X) + d\omega(X,IY,IZ) +\\
&\quad +\omega([IY,IZ],X) - g([IY,Z],X) - g([Y,IZ],X)\\
&= d\omega(X,Y,Z) + d\omega(X,IY,IZ) +\\
&\quad +g(-[Y,Z] - [IY,IZ] + I[IY,Z] + I[Y,IZ], IX)\\
&= d\omega(X,Y,Z) + d\omega(X,IY,IZ) - g(N_I(Y,Z),IX) = 0\,.
\end{aligned}
$$

□

Proposition 5. *Let (M,I,g) be a para-Kähler manifold. Then the eigendistributions $T^{\pm}M$ of I are g-isotropic. Their leaves are lagrangian submanifolds with respect to the para-Kähler form ω. In particular, g has split signature (n,n), where $n = \dim_C M$.*

Proof. Let $X \in T^{\pm}M$. Then, since I is an anti-isometry, we have

$$g(X,X) = -g(IX,IX) = -g(\pm X, \pm X) = -g(X,X)\,.$$

This shows that the I-invariant foliations $T^{\pm}M$ are g-isotropic and hence ω-lagrangian. □

Kähler manifolds can be characterized as riemannian manifolds with holonomy group in the unitary group. The corresponding characterization of para-Kähler manifolds is the following.

Proposition 6. *Let (M,g) be a (connected) pseudo-riemannian manifold of dimension $2n$. Then there exists a para-complex structure I on M such that (M,I,g) is a para-Kähler manifold if and only if the holonomy group of (M,g) is a subgroup of the para-unitary group, i.e., if there exists a point $p \in M$ and a linear isometry $T_pM \cong \mathbb{R}^{2n}$ which identifies the holonomy group $\mathrm{Hol}_p(M,g) \subset \mathrm{O}(T_pM, g_p)$ with a subgroup of the para-unitary group $\mathrm{U}^{\pi}(\mathbb{R}^{2n})$.*

Proof. If (M,I,g) is a para-Kähler manifold, $\dim_C M = n$, then

$$\mathrm{Hol}_p(M,g) \subset \mathrm{Aut}(T_pM, I_p, g_p) \cong \mathrm{U}^{\pi}(\mathbb{R}^{2n})\,.$$

Conversely, let (M,g) be a pseudo-riemannian manifold of dimension $2n$ such that $\phi \mathrm{Hol}_p \times (M,g)\phi^{-1} \subset \mathrm{U}^\pi(\mathbb{R}^{2n})$, where $\phi : T_pM \to \mathbb{R}^{2n}$ is a linear isometry. Then we can define a para-complex structure I_p on the vector space T_pM by

$$I_p := \phi^{-1} I_0 \phi\,,$$

where (I_0, g_0) denotes the standard para-hermitean structure on $\mathbb{R}^{2n}$. Since ϕ is an isometry and I_0 is an anti-isometry with respect to g_0, I_p is an anti-isometry with respect to g_p. As I_p is invariant under the holonomy group $\mathrm{Hol}_p(M,g)$, it extends (by parallel transport) to a parallel anti-isometric para-complex structure I on (M,g). Thus (M,I,g) is a para-Kähler manifold. □

The para-Kähler potential

Theorem 2. *Let (M,I,g) be a para-Kähler manifold with para-Kähler form ω. Then for any point $p \in M$ there exists a real-valued function K defined on some open neighborhood $U \subset M$ of p such that $\omega = \partial_-\partial_+ K$ on U. The function K is unique up to addition of a real-valued function f satisfying the equation $\partial_-\partial_+ f = 0$. Any such function is of the form $f = f_+ + f_-$, where $f_\pm : U \to \mathbb{R}$ satisfies $\partial_\mp f_\pm = 0$. Conversely, let (M,I) be a para-complex manifold and K a real-valued function defined on some open subset $U \subset M$ such that the two-form $\partial_-\partial_+ K$ is non-degenerate. Then $g := \omega(I\cdot,\cdot)$ is a pseudo-riemannian metric such that (U,I,g) is a para-Kähler manifold.*

Proof. Let $(z^i_\pm)$ be adapted coordinates defined on some open neighborhood U of $p \in M$, which map U onto the product of two simply connected open sets $U^\pm \subset \mathbb{R}^n$, $n = \dim_C M$. We will also assume that $z^i_\pm(p) = 0$. In particular, the first cohomology of U vanishes; $H^1(U,\mathbb{R}) = 0$. Therefore, since ω is closed, there exists a one-form θ on U such that $\omega = d\theta$. We decompose θ into its homogeneous components: $\theta = \theta^+ + \theta^-$, $\theta^+ \in \Omega^{1+,0-}(U)$, $\theta^- \in \Omega^{0+,1-}(U)$. This shows that

$$d\theta = \partial_+\theta^+ + (\partial_-\theta^+ + \partial_+\theta^-) + \partial_-\theta^-\,.$$

From the fact that ω is of type $(1,1)$ we obtain the equations:

$$\partial_\pm\theta^\pm = 0 \quad \text{and} \quad \partial_-\theta^+ + \partial_+\theta^- = \omega\,.$$

Lemma 1. *(Para-Dolbeault Lemma) Under the above topological assumptions on $U \subset M$, the equation $\partial_\pm\theta^\pm = 0$ implies the existence of a real-valued function $K^\pm$ such that*

$$\theta^\pm = \partial_\pm K^\pm\,.$$

The function $K^\pm$ is unique up to addition of a real-valued function $f_\mp$ which satisfies $\partial_\pm f_\mp = 0$.

Proof. The uniqueness statement is obvious. To prove the existence, suppose e.g. that $\partial_+\theta^+ = 0$ on $U \cong U^+ \times U^-$. Then we can define a function K^+ on $U^+ \times U^- \cong U$ by

$$K^+(z_+, z_-) := \int_{(0,z_-)}^{(z_+,z_-)} \theta^+\,,$$

where $z_\pm := (z_\pm^1, \dots, z_\pm^n)$ and the integration is over any path from $(0, z_-)$ to (z_+, z_-) contained in $U^+ \times \{z_-\}$. The condition $\partial_+ \theta^+ = 0$ ensures that the integral is path independent. In fact, it implies that the one-form $\theta^+|_{U^+\times\{z_-\}}$ is closed and hence exact, since U^+ is simply connected. □

In virtue of the lemma we can choose two real-valued functions $K^\pm$ such that $\partial_\pm K^\pm = \theta^\pm$. Putting $K := K^+ - K^-$, we obtain

$$\partial_-\partial_+ K = \partial_-\partial_+ K^+ + \partial_+\partial_- K^- = \partial_-\theta^+ + \partial_+\theta^- = \omega\,.$$

It is clear that the function K is unique up to adding a solution of $\partial_-\partial_+ f = 0$. We have to show that any solution is of the form $f = f_+ + f_-$, where $\partial_\mp f_\pm = 0$. Expanding

$$\partial_+ f = \sum f_i^+ dz_+^i\,, \qquad \text{with} \qquad f_i^+ := \frac{\partial f}{\partial z_+^i}\,,$$

we get

$$0 = \partial_-\partial_+ f = \sum \frac{\partial f_i^+}{\partial z_-^j} dz_-^j \wedge dz_+^i\,.$$

Thus $\partial f_i^+/\partial z_-^j = 0$ and the functions f_i^+ depend only on the 'positive' coordinates $z_+ = (z_+^1, \dots, z_+^n)$; $f_i^+ = f_i^+(z_+)$. This implies that

$$f(z_+, z_-) = \sum \int_0^{z_+} f_i^+(\zeta) d\zeta^i + f_-(z_-)\,,$$

where $\zeta = (\zeta^1, \dots, \zeta^n)$ and $f_-(z_-) = f(0, z_-)$ (the 'constant of integration') is a function of the 'negative' coordinates $z_- = (z_-^1, \dots, z_-^n)$ alone. The path integral is well defined since U^+ is simply connected. Setting $f_+(z_+) := \sum \int_0^{z_+} f_i^+(\zeta) d\zeta^i$ we obtain the desired decomposition $f(z_+, z_-) = f_+(z_+) + f_-(z_-)$.

Now we prove the converse. Let K be a real-valued function on some open subset $U \subset M$ such that $\omega := \partial_-\partial_+ K$ is a non-degenerate two-form. Since ω is automatically closed it is a symplectic structure. Moreover, it is of type $(1,1)$, which is equivalent to $I^*\omega = -\omega$. This implies that $g := \omega(I\cdot, \cdot)$ is symmetric,

$$g(X, Y) = \omega(IX, Y) = (I^*\omega)(X, IY) = -\omega(X, IY) = \omega(IY, X) = g(Y, X)\,,$$

and hence a pseudo-riemannian metric. Moreover, $I^* g = -g$, i.e., g is para-hermitean. Now the theorem follows from theorem 1. □

2.3 Affine special para-Kähler manifolds

Definition 9. *An* (affine) special para-Kähler manifold (M, I, g, ∇) *is a para-Kähler manifold* (M, I, g) *endowed with a flat torsion-free connection* ∇ *such that*

(i) ∇ *is symplectic, i.e.,* $\nabla\omega = 0$ *and*

(ii) ∇I *is a symmetric (1,2)-tensor field, i.e.,* $(\nabla_X I)Y = (\nabla_Y I)X$ *for all* X, Y.

We will not discuss *projective* special para-Kähler manifolds in this paper. Therefore, in the following, we will do without the adjective 'affine' and speak simply of *special para-Kähler manifolds*. Now we give an extrinsic construction of special para-Kähler manifolds from certain para-holomorphic immersions into the para-complex vector space $V = T^*C^n \cong C^{2n}$.

The cotangent bundle $N = T^*M$ of any para-complex manifold M carries a canonical non-degenerate exact C-valued two-form Ω of type $(2,0)$ which is para-holomorphic, in the sense that it defines a para-holomorphic section of the para-holomorphic vector bundle $\wedge^{2,0}T^*N$. We will now describe Ω explicitly. If $(z^1,\dots,z^n)$ are local para-holomorphic coordinates on $U \subset M$, then any point of $T_p^*M = \mathrm{Hom}_{\mathbb{R}}(T_pM,\mathbb{R}) \cong \mathrm{Hom}_C(T_pM,C)$, $p \in M$, is of the form $\sum w_i dz^i|_p$. Here $\mathrm{Hom}_C(T_pM,C)$ denotes the vector space of homomorphisms from the para-complex vector space (T_pM, I_p) into the para-complex vector space C. As usual, the z^i and w_j can be considered as locally defined (para-)holomorphic functions on T^*M. They form a system of para-holomorphic coordinates on the bundle $T^*M|_U$. The functions w_j induce a system of linear para-holomorphic coordinates on each fibre T_p^*M for all $p \in U$. In the para-holomorphic coordinates (z^i, w_j) the two-form Ω is given by

$$\Omega = \sum dz^i \wedge dw_i = -d\left(\sum w_i dz^i\right).$$

One can check that the one-form $\sum w_i dz^i$ does not depend on the choice of coordinates. Therefore, Ω is also coordinate independent. We will call it *the symplectic form* of T^*M.

In the following, V will always denote the para-holomorphic vector space $V = T^*C^n \cong C^{2n}$ endowed with its standard para-complex structure I_V, its symplectic form Ω and with the para-complex conjugation $\tau : V \to V$, $v \mapsto \tau(v) := \overline{v}$, whose fixed point set is $V^\tau := T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. We can choose a system of linear para-holomorphic coordinates (z^i) on C^n which are real-valued on the subspace $\mathbb{R}^n \subset C^n = \mathbb{R}^n \oplus e\mathbb{R}^n$. The corresponding para-holomorphic coordinates (z^i, w_j) on V are linear and real-valued on the subspace V^τ. The algebraic data (Ω, τ) on V induce a para-hermitean scalar product g_V on V by

$$g_V(v,w) := \mathrm{Re}\big(e\Omega(v,\tau(w))\big) \qquad \forall v, w \in V.$$

(V, I, g_V) is a flat para-Kähler manifold, whose para-Kähler form ω_V is given by

$$\omega_V(v,w) := g_V(I_V v, w) = \mathrm{Im}\big(e\Omega(v,\tau(w))\big) \qquad \forall v, w \in V.$$

Remark 1: the combination $\gamma_V := g_V + e\omega_V = e\Omega(\cdot, \tau\cdot)$ is a *para-hermitean form*, in the sense that it is C-sequilinear and $\gamma(w,v) = \overline{\gamma(v,w)}$ for all $v, w \in V$. It is obviously non-degenerate.

Let (M, I) be a (connected) para-complex manifold of para-complex dimension n.

Definition 10. *A para-holomorphic immersion $\phi : M \to V$ is called* para-kählerian *if $g := \phi^* g_V$ is non-degenerate and* lagrangian *if $\phi^*\Omega = 0$.*

The following proposition is an easy consequence of theorem 1, since the pull-back of a closed form is closed.

Proposition 7. *Any para-kählerian immersion $\phi : M \to V$ induces on M the structure of a para-Kähler manifold (M, I, g) with para-Kähler form $\omega = g(I\cdot, \cdot) = \phi^*\omega_V$.*

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Remark 2: for any para-Kähler manifold (M, I, g), with para-Kähler form ω, the combination $\gamma = g + e\omega$ is a field of non-degenerate para-hermitean forms. For a para-holomorphic immersion $\phi : M \to V$ the following conditions are equivalent:

(i) ϕ is para-kählerian, i.e., $g = \phi^* g_V$ is non-degenerate

(ii) $\omega = \phi^* \omega_V$ is non-degenerate and

(iii) $\phi^* \gamma_V$ is non-degenerate.

Under these assumptions, $\gamma = g + e\omega = \phi^* \gamma_V$.

Lemma 2. *Let $\phi : M \to V$ be a para-kählerian lagrangian immersion. Then the para-Kähler form $\omega = g(I\cdot, \cdot) = -g(\cdot, I\cdot)$ of M is given by*

$$\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i \,,$$

where $\tilde{x}^i := \mathrm{Re}\, \phi^* z^i$ and $\tilde{y}_i := \mathrm{Re}\, \phi^* w_i$.

Proof. Since $\Omega = \sum dz^i \wedge dw_i = \sum (dz^i \otimes dw_i - dw_i \otimes dz^i)$, the para-hermitean form $\gamma_V = e\Omega(\cdot, \tau\cdot)$ is given by

$$\gamma_V = e \sum (dz^i \otimes d\overline{w}_i - dw_i \otimes d\overline{z}^i) \,. \tag{2.1}$$

Decomposing the functions z^i and w_i into their real and imaginary parts,

$$z^i = x^i + eu^i \,, \qquad w_i = y_i + ev_i \,,$$

we obtain

$$\begin{aligned} \omega_V &= \mathrm{Im}\, \gamma_V = \frac{1}{2} \left(\sum (dz^i \otimes d\overline{w}_i - dw_i \otimes d\overline{z}^i) + \sum (d\overline{z}^i \otimes dw_i - d\overline{w}_i \otimes dz^i) \right) \\ &= \sum (dx^i \wedge dy_i - du^i \wedge dv_i) \,. \end{aligned}$$

On the other hand,

$$\mathrm{Re}\, \Omega = \sum (dx^i \wedge dy_i + du^i \wedge dv_i) \,.$$

From the assumption $\phi^* \Omega = 0$ we obtain

$$0 = \phi^* (\mathrm{Re}\, \Omega) = \phi^* \left(\sum (dx^i \wedge dy_i + du^i \wedge dv_i) \right)$$

and hence

$$\omega = \phi^* \omega_V = \phi^* \left(\sum (dx^i \wedge dy_i - du^i \wedge dv_i) \right) = 2\phi^* \left(\sum dx^i \wedge dy_i \right) = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i \,.$$

□

Corollary 1. *Let $\phi : M \to V$ be a para-kählerian lagrangian immersion. Then there exists a canonical flat torsion-free connection ∇ on M. It is characterized by the condition that $\nabla(\mathrm{Re}\, \phi^* df) = 0$ for all complex affine functions f on V.*

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Proof. The lemma implies that any point $p \in M$ has an open neighborhood U on which the functions $(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}_1, \ldots, \tilde{y}_n)$ form a system of local coordinates and, hence, define a flat torsion-free connection ∇^U on U. Obviously, $\nabla^{U_1} = \nabla^{U_2}$ on the overlap $U_1 \cap U_2$ of two such coordinate domains. Therefore, there exists a unique flat torsion-free connection ∇ on M which is the common extension of the locally defined connections ∇^U. It is characterized by the system of equations $\nabla(\mathrm{Re}\,\phi^* dz^i) = \nabla(\mathrm{Re}\,\phi^* dw_i) = 0$, $i = 1, 2, \ldots, n$. □

Theorem 3. *Let $\phi : M \to V$ be a para-kählerian lagrangian immersion with induced geometric data (I, g, ∇). Then (M, I, g, ∇) is a special para-Kähler manifold. Conversely, any simply connected special para-Kähler manifold (M, I, g, ∇) admits a para-kählerian lagrangian immersion inducing the special geometric data (I, g, ∇) on M. The para-kählerian lagrangian immersion ϕ is unique up to an affine transformation of V whose linear part belongs to the group $\mathrm{Aut}_C(V, \Omega, \tau) = \mathrm{Aut}_{\mathbb{R}}(V, I_V, \Omega, \tau) = \mathrm{Sp}(\mathbb{R}^{2n})$.*

Proof. Let $\phi : M \to V$ be a para-kählerian lagrangian immersion with induced geometric data (I, g, ∇). We have to check that i) $\nabla\omega = 0$ and that ii) ∇I is symmetric. The first condition is satisfied since ω has constant coefficients with respect to the ∇-affine local coordinates $(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}_1, \ldots, \tilde{y}_n)$. In order to verify the second condition, we evaluate a ∇-parallel 1-form ξ on the alternation of ∇I:

$$\xi((\nabla_X I)Y - (\nabla_Y I)X) = (\nabla_X \xi \circ I)Y - (\nabla_Y \xi \circ I)X = d(\xi \circ I)(X, Y)$$

for all vector fields X and Y on M. Here we have used that ∇ is torsion-free. Since ∇ is flat it is sufficient to check that the one-form $\xi \circ I$ is closed for all ∇-parallel 1-forms ξ. Any ∇-parallel 1-form is a linear combination with constant coefficients of the differentials $d\tilde{x}^i$ and $d\tilde{y}_i$. So we have to show that the one-forms $d\tilde{x}^i \circ I$ and $d\tilde{y}_i \circ I$ are closed.

The functions z^i and w_i on V are para-holomorphic. Since the map $\phi : M \to V$ is para-holomorphic, this implies that the one-forms $\phi^* dz^i$ and $\phi^* dw_i$ on M are of type $(1, 0)$. Thus

$$\begin{aligned} \phi^* dz^i &= d\tilde{x}^i + e d\tilde{x}^i \circ I \\ \phi^* dw_i &= d\tilde{y}_i + e d\tilde{y}_i \circ I\,. \end{aligned}$$

These equations show that the one-forms $d\tilde{x}^i \circ I$ and $d\tilde{y}_i \circ I$ are closed, which proves ii).

Now we prove the converse statement. Let (M, I, g, ∇) be a simply connected special para-Kähler manifold. We have to construct a para-kählerian lagrangian immersion $\phi : M \to V$, which induces the special geometric structures on M. Since the symplectic structure ω is ∇-parallel, for any point $p \in M$ there exists a system of local ∇-affine coordinates $(\tilde{x}^1, \ldots, \tilde{x}^n, \tilde{y}_1, \ldots, \tilde{y}_n)$ defined on some simply connected open neighborhood $U \subset M$ of p such that $\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i$. The coordinate system near p is unique up to an affine symplectic transformation. We define $2n$ one-forms ω^i and η_i of type $(1, 0)$ by

$$\begin{aligned} \omega^i &= d\tilde{x}^i + e d\tilde{x}^i \circ I \\ \eta_i &= d\tilde{y}_i + e d\tilde{y}_i \circ I\,. \end{aligned}$$

The symmetry of ∇I implies that these forms are closed, by the same calculation as above. It follows that ω^i and η_i are closed para-holomorphic sections of $\wedge^{1,0}T^*M$. As U is simply connected, there exists para-holomorphic functions $\tilde{z}^i$ and $\tilde{w}_i$ defined at $q \in U$ by

$$\tilde{z}^i(q) := \tilde{x}^i(p) + \int_p^q \omega^i ,$$
$$\tilde{w}_i(q) := \tilde{y}_i(p) + \int_p^q \eta_i ,$$

where the integral is over any path $c \subset U$ from p to q. Thus $\omega^i = d\tilde{z}^i$, $\eta_i = d\tilde{w}_i$ and $\tilde{x}^i = \operatorname{Re} \tilde{z}^i$, $\tilde{y}_i = \operatorname{Re} \tilde{w}_i$.

Now we can define a para-holomorphic map $\phi_U : U \to V \cong C^{2n}$ by the equations

$$\phi_U^* z^i := \tilde{z}^i \qquad \text{and} \qquad \phi_U^* w_i := \tilde{w}_i .$$

We claim that $\phi_U : (U, I, g, \nabla) \to V$ is a para-kählerian lagrangian immersion with induced geometric data (g, ∇). To see that ϕ_U is an immersion (and even an embedding), it suffices to remark that $\psi_U := \operatorname{Re} \phi_U : U \to V^\tau \cong \mathbb{R}^{2n}$ is given by

$$\psi_U^* x^i = \tilde{x}^i \qquad \text{and} \qquad \psi_U^* y_i = \tilde{y}_i ,$$

which is obviously a diffeomorphism of U onto its image. Using the formulas $\Omega = \sum dz^i \wedge dw_i$, $\phi_U^* z^i = d\tilde{x}^i + e d\tilde{x}^i \circ I$, $\phi^* dw_i = d\tilde{y}_i + e d\tilde{y}_i \circ I$ and $\omega = 2 \sum d\tilde{x}^i \wedge d\tilde{y}_i$ we calculate

$$\phi_U^* \Omega = \frac{1}{2}(\omega + \omega(I\cdot, I\cdot)) + \frac{1}{2}(\omega(I\cdot, \cdot) + \omega(\cdot, I\cdot)) = 0 ,$$

which shows that ϕ_U is lagrangian. Similarly, using the formulas $\omega_V = (dx^i \wedge dy_i - du^i \wedge dv_i)$, $\phi_U^* dx^i = d\tilde{x}^i$, $\phi_U^* du^i = d\tilde{x}^i \circ I$, $\phi_U^* dy_i = d\tilde{y}_i$ and $\phi_U^* dv_i = d\tilde{y}_i \circ I$, we calculate

$$\phi_U^* \omega_V = \sum (d\tilde{x}^i \wedge d\tilde{y}_i - (d\tilde{x}^i \circ I) \wedge (d\tilde{y} \circ I)) = \frac{1}{2}(\omega - I^* \omega) = \omega .$$

This shows that the immersion ϕ_U is para-kählerian with para-Kähler form $\phi_U^* \omega_V = \omega$ and para-Kähler metric $\phi_U^* g_V = g$. The flat torsion-free connection induced by the para-kählerian lagrangian immersion ϕ_U coincides with the special connection ∇, since the functions $\operatorname{Re} \phi_U^* z^i = \tilde{x}^i$ and $\operatorname{Re} \phi_U^* w_i = \tilde{y}_i$ form a system of ∇-affine coordinates on U. It is clear that the germ ϕ_p of the para-kählerian lagrangian immersion ϕ_U at p is unique up to an affine transformation of V with linear part in $\mathrm{Sp}(\mathbb{R}^{2n})$ and that this germ has a unique continuation $\phi_{c(t)}$ along any path $c : [0,1] \to M$ from p to q. Since M is simply connected, the value $\phi_q := \phi_{c(1)}$ of the above continuation at the endpoint is independent of c. Therefore, the para-kählerian lagrangian immersion $\phi_U : U \to V$ extends uniquely to a para-kählerian lagrangian immersion $\phi : M \to V$ with the claimed properties. $\square$

The theorem implies that any special para-Kähler manifold can be locally constructed from a para-holomorphic prepotential, as we shall explain now. For any para-holomorphic function $F : U \to C$ on an open subset $U \subset C^n$ we can consider the para-holomorphic Hessian

$$\partial^2 F := \left(\frac{\partial^2 F}{\partial z^i \partial z^j} \right) .$$

Corollary 2. *Let $F : U \to C$ be a para-holomorphic function defined on an open subset $U \subset C^n$ which satisfies the non-degeneracy condition* $\det(\operatorname{Im}\partial^2 F) \neq 0$*. Then $\phi_F := dF : U \to T^*C^n = V$ is a para-kählerian lagrangian immersion and hence defines a special para-Kähler manifold $M_F = (U, I, g, \nabla)$. Conversely, any special para-Kähler manifold is locally isomorphic to a manifold of the form M_F.*

Proof. It is clear that $\phi_F : U \to V$ is a para-holomorphic lagrangian immersion. In fact, its components

$$\phi_F^* z^i = z^i \qquad \text{and} \qquad \phi_F^* w_i = \frac{\partial F}{\partial z^i}$$

are para-holomorphic functions and

$$\phi_F^* \Omega = \phi_F^* \left(-d \sum w_i dz^i \right) = -d\phi_F^* \left(\sum w_i dz^i \right) = -ddF = 0\,.$$

Let us check, with the aid of equation (2.1), that the non-degeneracy condition implies that ϕ_F is para-kählerian:

$$\begin{aligned} \phi_F^* \gamma_V &= e \sum \left(dz^i \otimes d\left(\frac{\partial \overline{F}}{\partial \overline{z}^i}\right) - d\left(\frac{\partial F}{\partial z^i}\right) \otimes d\overline{z}^i \right) \\ &= e \sum \left(\frac{\partial^2 \overline{F}}{\partial \overline{z}^i \partial \overline{z}^j} - \frac{\partial^2 F}{\partial z^i \partial z^j} \right) dz^i \otimes d\overline{z}^j \qquad (2.2) \\ &= -2 \sum \left(\operatorname{Im} \frac{\partial^2 F}{\partial z^i \partial z^j} \right) dz^i \otimes d\overline{z}^j\,. \qquad (2.3) \end{aligned}$$

This shows that ϕ_F is a para-kählerian lagrangian immersion.

Conversely, by theorem 3, we know that any special para-Kähler manifold is locally defined by a para-kählerian lagrangian immersion $\phi_U : U \to V$, where U is (biholomorphic to) a simply connected open subset of C^n. Restricting U and composing ϕ_U with an affine transformation of V whose linear part is in $\mathrm{Sp}(\mathbb{R}^{2n})$, if necessary, we can assume that the functions z^i on V induces a global system of para-holomorphic coordinates $(\phi_U^* z^i)$ on U. (Equivalently, we can choose the canonical coordinates (z^i, w_j) of V adapted to the immersion.) Then we can write the lagrangian submanifold $\phi_U(U) \subset V$ as a graph: $w_i = F_i(z)$, $i = 1, 2, \ldots, n$. The lagrangian condition $\phi_U^* \Omega = \sum dz^i \wedge dF_i = 0$ is equivalent to the para-holomorphic one-form $\sum F_i dz^i$ on U being closed and hence exact, since U is simply connected. This proves the existence of a para-holomorphic function $F : U \to C$ such that $\phi_U = \phi_F$. Finally, by the previous calculation, the kählerian condition on the holomorphic immersion $\phi_U = \phi_F$ is equivalent to $\det(\operatorname{Im}\partial^2 F) \neq 0$. □

Remark 3: the above construction can be easily reformulated in purely real terms without using para-complex numbers. In fact, an immersion $\phi : M \to V$ of a real differentiable manifold of dimension $2n = \frac{1}{2}\dim_{\mathbb{R}} V$ satisfies $\phi^*\Omega = 0$ if and only if it is lagrangian with respect to the two real symplectic structures

$$\Omega_{\pm} = \frac{1}{2} \sum (dz_+^i \wedge dw_i^+ \pm dz_-^i \wedge dw_i^-)\,,$$

where $z^i_{\pm} = x^i \pm u^i$ and $w^{\pm}_i = y_i \pm v_i$. The condition $\phi^*\Omega = 0$ implies immediately that ϕ induces a para-complex structure on M such that $\phi : M \to V$ is para-holomorphic. However, it seems more natural to us to describe para-complex manifolds using para-complex numbers.

3. Fermions and supersymmetry in 5 and 4 dimensions

In the following section we review the relevant properties of Clifford algebras, fermions and supersymmetry algebras. We start in section 3.1 with those properties which can be discussed in arbitrary dimension and signature. In section 3.2 we specialize to 5 dimensions. We derive realizations of the minimal supersymmetry algebra in terms of Dirac and of symplectic Majorana spinors and show that the R-symmetry group is SU(2). In section 3.3 we obtain supersymmetry algebras in dimensions $(1,3)$ and $(0,4)$ by dimensional reduction over space and time, respectively. The R-symmetry group now contains an additional abelian factor, which is U(1) for Minkowski, but SO(1,1) for euclidean space-time signature.

Our discussion of Clifford algebras and spinors combines [41, 40, 42], where more details can be found. This section should be readable for physicists as well as for mathematicians. Note that in this section and in [41, 42] spinors are taken to be commuting (complex-valued) objects, whereas in sections 4 and 5, and in [40] anticommuting (Grassmann-valued) spinor fields[12] are used, as usual in supersymmetric field theories.

3.1 Clifford algebras, spinors and supersymmetry in (t, s) dimensions

Let us first consider space-times with t time-like and s space-like dimensions. Space-time indices will be denoted by $\mu, \nu, \ldots$ More precisely, we consider the vector space $V := \mathbb{R}^n$, $n = t + s$, endowed with the pseudo-euclidean scalar product η given by $\eta_{\mu\nu} := \eta(e_\mu, e_\nu) = \mathrm{diag}(-1, \ldots, -1, +1, \ldots, +1)$, with respect to the standard basis (e_μ) of $\mathbb{R}^n$. The negative (positive) values correspond to time-like (space-like) directions. We will say that such a space-time is (t, s)–dimensional.

The Clifford algebra

The Clifford algebra $Cl_{t,s}$ can be defined as the algebra generated by an orthonormal basis (e^μ) of the dual vector space V^* subject to the defining relations[13]

$$e^\mu e^\nu + e^\nu e^\mu = 2\,\eta^{\mu\nu}1\,, \tag{3.1}$$

where $\eta^{\mu\nu}$ is the inverse of $\eta_{\mu\nu}$ and 1 is the unit in the algebra. The gamma-matrices $\gamma^\mu = (\gamma^{\mu}{}_{\alpha}{}^{\beta})$ are the matrices which represent the generators e^μ of $Cl_{t,s}$ in an irreducible

[12] See subsection 3.4 for a mathematical definition.

[13] In the mathematical literature one often finds a minus sign on the r.h.s. of equation (3.1). If one simultaneously defines the space-time metric with a minus sign relative to our choice, as for example in [61], then $Cl_{t,s}$ denotes the same algebra as in the present paper. There are, however, also authors who denote our $Cl_{t,s}$ by $Cl_{s,t}$.

representation of the complex Clifford algebra $\mathbb{C}l_n = Cl_{t,s} \otimes \mathbb{C}$. They are complex square matrices of size $2^{[n/2]}$ satisfying the Clifford relations

$$\{\gamma^\mu, \gamma^\nu\} := \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\,\eta^{\mu\nu}\,\mathbb{1}\,, \tag{3.2}$$

where $\mathbb{1}$ is the identity map $\mathbb{C}^{2^{[n/2]}} \to \mathbb{C}^{2^{[n/2]}}$.

The spin group and the spinor module

The group $\mathrm{Spin}(t,s) \subset Cl_{t,s}$ which consists of products of an even number of unit vectors (i.e. $v \in V^*$ such that $\eta(v,v) = \pm 1$) is called the *spin group*. The maximal connected subgroup $\mathrm{Spin}_0(t,s) \subset \mathrm{Spin}(t,s)$ is called the *connected spin group*. The $\mathrm{Spin}(t,s)$-representation which is obtained by restricting an irreducible $\mathbb{C}l_n$-representation to $\mathrm{Spin}(t,s) \subset \mathbb{C}l_n$, is called *the complex spinor representation*. The corresponding representation space $\mathbb{S}$ is called *the complex spinor module* and its elements $\lambda = (\lambda_\alpha)$ are called *Dirac spinors*. The complex spinor module is an irreducible complex module if the space-time dimension $n = t+s$ is odd. It is the sum of two irreducible complex modules (*Weyl spinors*) $\mathbb{S}_+$ and $\mathbb{S}_-$ if $n = t+s$ is even.

The Lie algebra $\mathrm{spin}(t,s) \subset Cl_{t,s}$ of the spin group is generated by the commutators $[e^\mu, e^\nu] = e^\mu e^\nu - e^\nu e^\mu$, which are represented by the matrices

$$2\gamma^{\mu\nu} := 2\gamma^{[\mu}\gamma^{\nu]} := \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = [\gamma^\mu, \gamma^\nu]\,. \tag{3.3}$$

This Lie algebra is canonically isomorphic to the Lorentz Lie algebra $\mathrm{so}(t,s) = \mathit{Lie}\,\mathrm{SO}(t,s)$. In fact, there is a canonical double covering of Lie groups $R : \mathrm{Spin}(t,s) \to \mathrm{SO}(t,s)$, $g \mapsto R(g)$, given by $R(g)v = gvg^{-1} \in \mathbb{R}^n$, for all $v \in \mathbb{R}^n$, where the product is multiplication in the Clifford algebra. We recall that, by definition, *the real spinor module* S of $\mathrm{Spin}(t,s)$ is the restriction of an irreducible module of the real Clifford algebra $Cl_{t,s}$ to $\mathrm{Spin}(t,s) \subset Cl_{t,s}$. As a $\mathrm{Spin}(t,s)$-module it is either irreducible or a sum of two irreducible *semi-spinor* modules S_+ and S_- (which may happen to be equivalent).

The Dirac conjugate and the corresponding invariant sesquilinear form on $\mathbb{S}$

Following [40] we take the space-like and time-like gamma matrices to be hermitean and anti-hermitean, respectively. One can then always find a matrix $A = (A^{\alpha\beta})$ with the property

$$(\gamma^\mu)^\dagger = (-1)^t A\gamma^\mu A^{-1}\,. \tag{3.4}$$

An explicit realization of A is

$$A = \prod_\tau \gamma_\tau\,, \tag{3.5}$$

where $\gamma_\mu := \eta_{\mu\nu}\gamma^\nu$. The product is over all time-like γ-matrices. It is taken in lexicographic order.

The above equation implies that the matrix A represents an isomorphism of $\mathrm{spin}(t,s)$-modules from the complex spinor module (3.3) to the complex-conjugate of its dual module (in which the generators $[e^\mu, e^\nu]$ of the Lie algebra $\mathrm{spin}(t,s)$ act by the matrices $-2(\gamma^{\mu\nu})^\dagger$). The matrix A features in the definition of the *Dirac conjugate* spinor

$$\overline{\lambda}^{(D)} := \lambda^\dagger A\,. \tag{3.6}$$

The map

$$\mathbb{S}\times\mathbb{S}\ni(\lambda,\chi)\mapsto\overline{\lambda}^{(D)}\chi=\lambda^\dagger A\chi=\lambda^*_\alpha A^{\alpha\beta}\chi_\beta\in\mathbb{C} \tag{3.7}$$

defines a non-degenerate $\mathrm{Spin}_0(t,s)$-invariant sesquilinear form $h_{\mathbb{S}}$ on $\mathbb{S}$, i.e., it is antilinear in λ and linear in χ. Moreover, it is either hermitean or skew-hermitean depending on the value of t. This follows from the fact that

$$A^\dagger=(-1)^{\frac{t(t+1)}{2}}A\,. \tag{3.8}$$

The Majorana conjugate and the corresponding invariant bilinear form C on $\mathbb{S}$. Another operation considered in the physical literature is the *Majorana conjugation*

$$\lambda\mapsto\overline{\lambda}:=\lambda^T\mathsf{C}\,. \tag{3.9}$$

The *charge conjugation matrix* $\mathsf{C}=(\mathsf{C}^{\alpha\beta})$ satisfies

$$\mathsf{C}^{\mathrm{T}}=\sigma\,\mathsf{C}\,,\qquad\gamma^{\mu\,\mathrm{T}}=\tau\,\mathsf{C}\gamma^\mu\mathsf{C}^{-1}\,, \tag{3.10}$$

where σ and τ can take the values ±1, depending on the dimension and signature of space-time and on the choice of C, see [41, 40] for a list of the possible values σ and τ.[14] The Majorana conjugation relates the complex spinor representation (3.3) to its dual representation; it represents an isomorphism of $\mathrm{spin}(t,s)$-modules. It can be considered as a non-degenerate complex bilinear form C

$$\mathbb{S}\times\mathbb{S}\ni(\lambda,\chi)\mapsto\overline{\lambda}\chi=\lambda^T\mathsf{C}\chi=\lambda_\alpha\mathsf{C}^{\alpha\beta}\chi_\beta\in\mathbb{C} \tag{3.11}$$

on the complex spinor module. The Majorana conjugation is invariant under the complex spin group $\mathrm{Spin}(n,\mathbb{C})\subset\mathbb{C}l_n$, $n=t+s$, which is the connected Lie group consisting of even products of unit vectors $v\in V^*\otimes\mathbb{C}$, $\eta^{\mathbb{C}}(v,v)=1$, with respect to the complexified scalar product $\eta^{\mathbb{C}}$. Note that $\mathrm{Spin}(n,\mathbb{C})\supset\mathrm{Spin}_0(t,s)$.

All bilinear forms on the (real or complex) spinor module invariant under the connected (real or complex) spinor group were described in [41], where the notion of an *admissible* bilinear form on the spinor module was introduced. For such a form β two fundamental invariants, which take the values ±1, are defined: the *symmetry* $\sigma(\beta)$ and and the *type* $\tau(\beta)$. In the case of $\beta=C$ these invariants correspond precisely to σ and τ in (3.10). In general, a bilinear form β is symmetric if $\sigma(\beta)=+1$ and skew-symmetric if $\sigma(\beta)=-1$. The operators γ^μ are symmetric with respect to β if $\tau(\beta)=+1$ and skew-symmetric if $\tau(\beta)=-1$. A bilinear form β for which the invariants $\sigma(\beta)$ and $\tau(\beta)$ are defined is automatically $\mathrm{Spin}_0(t,s)$-invariant, as was shown in [41]. So admissibility implies $\mathrm{Spin}_0(t,s)$-invariance.

[14] We use here the notation of [41], explained below, where σ stands for symmetry and τ for type, which correspond to $-\varepsilon$ and $-\eta$ in [40].

A basis of the vector space of $\mathrm{Spin}_0(t,s)$-invariant bilinear forms on the spinor module which consists of non-degenerate admissible forms was constructed in [41] for all (t,s). For even space-time dimension, the vector space of $\mathrm{Spin}_0(t,s)$-invariant (and hence $\mathrm{Spin}(n,\mathbb{C})$-invariant) complex bilinear forms on the complex spinor module is two-dimensional. This corresponds to the fact that there are two independent matrices C which satisfy (3.10) with different values of σ and τ. An explicit example can be found in the appendix.

Majorana and symplectic Majorana spinors

In many physical applications one uses Weyl or Majorana spinors instead of Dirac spinors. Weyl spinors exist if the space-time dimension is even, $n = t + s$, as already discussed above. They are obtained from Dirac spinors by imposing a chirality constraint, and we will come back to them in section 3.3. Majorana spinors exist if one can impose a reality constraint. If $t - s = 0, 1, 2, 6$ and $7 \pmod 8$ one can impose a reality constraint on a single Dirac spinor, leading to Majorana spinors and for $t - s = 0 \pmod 8$ also to Majorana-Weyl spinors. In mathematical terms this means that the complex spinor representation admits a $\mathrm{Spin}(t,s)$-invariant real structure (i.e., antilinear involution). In the remaining cases, $t - s = 3, 4$ and $5 \pmod 8$, the complex spinor representation admits a $\mathrm{Spin}(t,s)$-invariant quaternionic structure (i.e., antilinear map which squares to $-\mathbb{1}$), leading to symplectic Majorana spinors and for $t - s = 4 \pmod 8$ also to symplectic Majorana-Weyl spinors. See [42] for a proof of these facts, which are summarized in table 1 of that paper. As we will use symplectic Majorana spinors to construct supersymmetry algebras and supersymmetric actions, we will give a detailed account in the following.

Reality constraints are formulated using $B := (\mathsf{C}A^{-1})^T$. As a consequence of (3.4) and (3.10) this matrix satisfies

$$(\gamma^\mu)^* = \tau(-1)^t B \gamma^\mu B^{-1} \,. \tag{3.12}$$

One can also show that

$$B^\dagger B = \mathbb{1} \qquad \text{and} \qquad B^* B = \pm \mathbb{1} \,. \tag{3.13}$$

Depending on the sign in the last equation, one can define Majorana or symplectic Majorana spinors.[15]

If $B^* B = \mathbb{1}$, the map $\lambda \mapsto B^* \lambda^*$ is a $\mathrm{Spin}(t,s)$-invariant real structure on the complex spinor module. Its real points satisfy the Majorana condition

$$\lambda^* = B \lambda \,, \tag{3.14}$$

which can also be expressed in the form $\overline{\lambda}^{(D)} = \overline{\lambda}$ where $\overline{\lambda}^{(D)}$ is the Dirac conjugate and $\overline{\lambda}$ is the Majorana conjugate.

In the case $B^* B = -\mathbb{1}$, the map[16]

$$\lambda \mapsto j_{\mathbb{S}} \lambda := -B^* \lambda^* \tag{3.15}$$

[15]Note that the condition $t - s = 3, 4, 5 \pmod 8$ is sufficient, but not necessary for the existence of symplectic Majorana spinors. In some cases, including $(t,s) = (1,3)$, both Majorana and symplectic Majorana spinors exist. See the appendix for more details.

[16]The sign has been chosen in order to be consistent with the conventions of [40].

is a Spin(t,s)-invariant quaternionic structure $j_{\mathbb{S}}$ on the complex spinor module $\mathbb{S}$. The invariance is a consequence of the equation

$$j_{\mathbb{S}}(\gamma^\mu \lambda) = \tau(-1)^t \gamma^\mu j_{\mathbb{S}}(\lambda)\,, \tag{3.16}$$

which is equivalent to (3.12). Notice that the quaternionic structure $j_{\mathbb{S}}$ is invariant under the group Pin$(t,s) \subset Cl_{t,s}$ generated by unit vectors $v \in V^*$, $\eta(v,v) = \pm 1$, if $\tau(-1)^t = +1$. In fact, $j_{\mathbb{S}}$ commutes with the operators γ^μ if $\tau(-1)^t = +1$ and anti-commutes with the γ^μ if $\tau(-1)^t = -1$, which is the case e.g. for $(t,s) = (1,4)$.

If $B^*B = -\mathbb{1}$, one can impose a so-called symplectic Majorana constraint on a pair of Dirac spinors λ^i, $i = 1,2$, which reads

$$(\lambda^i)^* = -B\lambda^j \epsilon_{ji}\,. \tag{3.17}$$

Here, ϵ_{ij} is an antisymmetric two-by-two matrix, which we take to be

$$(\epsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.18}$$

The indices $i,j,\ldots = 1,2$ are raised and lowered according to the so-called NW-SE convention: $\lambda_i := \lambda^j \epsilon_{ji}$ and $\lambda^i = \epsilon^{ij}\lambda_j$, where $\epsilon^{ij} = \epsilon_{ij}$ and therefore $\epsilon^{ik}\epsilon_{kj} = -\delta^i_j$.

In this formalism one uses double spinors $\lambda = (\lambda^1, \lambda^2)^T$ which are elements of the complex Spin(t,s)-module

$$\mathbb{S} \otimes \mathbb{C}^2 = \mathbb{S} \otimes_{\mathbb{C}} \mathbb{C}^2 \cong \mathbb{S} \oplus \mathbb{S}\,.$$

The tensor product of the quaternionic structure $j_{\mathbb{S}}$ of $\mathbb{S}$ with the standard quaternionic structure $j_{\mathbb{C}^2}$ of $\mathbb{C}^2 = \mathbb{H}$,

$$j_{\mathbb{C}^2}(z^1 \vec{e}_1 + z^2 \vec{e}_2) = \overline{z}^1 \vec{e}_2 - \overline{z}^2 \vec{e}_1\,,$$

defines a Spin(t,s)-invariant real structure

$$\rho = j_{\mathbb{S}} \otimes j_{\mathbb{C}^2} \tag{3.19}$$

on $\mathbb{S} \otimes \mathbb{C}^2$. The real points of $\mathbb{S} \otimes \mathbb{C}^2$ are exactly the symplectic Majorana spinors defined by (3.17):

$$\rho\left[\begin{pmatrix} \lambda^1 \\ \lambda^2 \end{pmatrix}\right] = \begin{pmatrix} -j_{\mathbb{S}}(\lambda^2) \\ j_{\mathbb{S}}(\lambda^1) \end{pmatrix} = \begin{pmatrix} -B^*(\lambda^i)^*\epsilon_{i1} \\ -B^*(\lambda^i)^*\epsilon_{i2} \end{pmatrix}. \tag{3.20}$$

Symplectic Majorana spinors form a real Spin(t,s)-submodule $\mathbb{S}_{SM} \subset \mathbb{S} \otimes \mathbb{C}^2$, isomorphic to the complex spinor module $\mathbb{S}$. In fact, $\rho(\lambda) = \lambda$ if and only if $\lambda^2 = j_{\mathbb{S}}(\lambda^1)$. Thus under the above isomorphism $\mathbb{S} \otimes \mathbb{C}^2 \cong \mathbb{S} \oplus \mathbb{S}$, $\mathbb{S}_{SM}$ corresponds to the graph $\{(\psi, j_{\mathbb{S}}(\psi) | \psi \in \mathbb{S}\}$ of the quaternionic structure $j_{\mathbb{S}}$ on $\mathbb{S}$.

It has often advantages to rewrite Dirac spinors as symplectic Majorana spinors, i.e., as pairs of Dirac spinors subject to the symplectic Majorana condition. For example, as we will see below, the R-symmetries of 5- and 4-dimensional supersymmetry algebras only act on the internal space $\mathbb{C}^2$.

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The induced bilinear form on the space of double spinors and its restriction to $\mathbb{S}_{SM}$

The tensor product

$$b = C \otimes \varepsilon \tag{3.21}$$

of the bilinear form C on $\mathbb{S}$ introduced in (3.11) and the antisymmetric bilinear form $\varepsilon(z, w) = z^i w^j \epsilon_{ji}$ on $\mathbb{C}^2$ defines a non-degenerate bilinear form on the space $\mathbb{S} \otimes \mathbb{C}^2$ of double spinors:

$$(\lambda, \chi) \mapsto \overline{\lambda}\chi := \overline{\lambda}^i \chi_i = \overline{\lambda}^i \chi^j \epsilon_{ji} = (\lambda^i)^T \mathsf{C} \chi^j \epsilon_{ji} = \lambda^i_\alpha \chi^j_\beta \mathsf{C}^{\alpha\beta} \epsilon_{ji} \,. \tag{3.22}$$

Here $\lambda = (\lambda^i), \chi = (\chi^i) \in \mathbb{S} \otimes \mathbb{C}^2$. Without imposing the reality constraint (3.17), this bilinear form is obviously invariant under the natural action of the group $\mathrm{Spin}(n, \mathbb{C}) \otimes \mathrm{Sp}(2, \mathbb{C})$, thus motivating the name symplectic Majorana spinor. It is symmetric if C is skew-symmetric and skew-symmetric if C is symmetric. The group $\mathrm{Sp}(2, \mathbb{C})$ is precisely the centralizer of $\mathrm{Spin}(n, \mathbb{C})$ in the group of automorphisms of the complex bilinear form b. Either the restriction of b or that of ib to the subspace of symplectic Majorana spinors $\mathbb{S}_{SM} \subset \mathbb{S} \otimes \mathbb{C}^2$ is a real valued bilinear form on $\mathbb{S}_{SM}$. This follows from the fact that $\mathbb{S}_{SM}$ is the set of fixed points of $\rho = j_{\mathbb{S}} \otimes j_{\mathbb{C}^2}$ and the following equations

$$j_{\mathbb{S}}^* C = \pm \overline{C} \,, \qquad j_{\mathbb{C}^2}^* \varepsilon = \overline{\varepsilon} \,, \qquad \rho^* b = \pm \overline{b} \,, \tag{3.23}$$

where

$$(j_{\mathbb{S}}^* C)(\lambda, \chi) := C(j_{\mathbb{S}}\lambda, j_{\mathbb{S}}\chi) \qquad \text{and} \qquad \overline{C}(\lambda, \chi) := \overline{C(\lambda, \chi)} \,.$$

The first equation of (3.23) is equivalent to

$$B^\dagger C B^* = \pm C^* \,, \tag{3.24}$$

which follows from (3.8) using the definition of B and the property that $B^* B = -1$. The sign $\pm$ on the right hand side of equations (3.23) and (3.24) equals $-\sigma(-1)^{t(t+1)/2}$, which is -1 for $(t, s) = (1, 4)$, for example. Obviously, the restriction of b to $\mathbb{S}_{SM}$ is $\mathrm{Spin}_0(t, s) \otimes \mathrm{SU}(2)$-invariant. Here $\mathrm{SU}(2)$ arises as the intersection $\mathrm{SU}(2) = \mathrm{Sp}(2, \mathbb{C}) \cap \mathrm{GL}(1, \mathbb{H})$, the subgroup of $\mathrm{Sp}(2, \mathbb{C})$, which preserves the symplectic Majorana constraint. The last equation of (3.23) shows that the restriction of b to symplectic Majorana spinors is either real or imaginary.

The induced sesquilinear form on the space of double spinors and its restriction to $\mathbb{S}_{SM}$

Similarly, the tensor product

$$h_{\mathbb{S} \otimes \mathbb{C}^2} = h_{\mathbb{S}} \otimes h_{\mathbb{C}^2}$$

of the $\mathrm{Spin}_0(t, s)$-invariant sesquilinear form $h_{\mathbb{S}}$ on $\mathbb{S}$ defined in (3.7) and the standard sesquilinear form $h_{\mathbb{C}^2}$ on $\mathbb{C}^2$ represented by (δ_{ij}) is a a non-degenerate sesquilinear form on $\mathbb{S} \otimes \mathbb{C}^2$. It is given by:

$$(\lambda, \chi) \mapsto \sum_{i=1}^{2} (\lambda^i)^\dagger A \chi^i = \sum_{i,j=1}^{2} (\lambda^i)^\dagger A \chi^j \delta_{ij} \,, \tag{3.25}$$

where $\lambda = (\lambda^i)$ and $\chi = (\chi^i)$ are elements of $\mathbb{S} \otimes \mathbb{C}^2$. This form is hermitean if A is hermitean and skew-hermitean if A is skew-hermitean, see (3.8). It is obviously invariant under the group $\mathrm{Spin}_0(t,s) \otimes \mathrm{U}(2)$. Moreover, depending on (t,s), the sesquilinear form is real or purely imaginary on the subspace $\mathbb{S}_{SM} \subset \mathbb{S} \otimes \mathbb{C}^2$ of symplectic Majorana spinors. This follows from

$$j_{\mathbb{S}}^* h_{\mathbb{S}} = \pm \overline{h}_{\mathbb{S}} \qquad \text{and} \qquad j_{\mathbb{C}^2}^* h_{\mathbb{C}^2} = \overline{h}_{\mathbb{C}^2}\,, \tag{3.26}$$

which is a consequence of (3.23) (with the same sign $\pm$), since $h_{\mathbb{S}} = C(j_{\mathbb{S}}\cdot,\cdot)$ and $h_{\mathbb{C}^2} = \varepsilon(j_{\mathbb{C}^2}\cdot,\cdot)$. This shows also that

$$h_{\mathbb{S}\otimes\mathbb{C}^2} = b(\rho\cdot,\cdot) \qquad \text{and} \qquad \rho^* h_{\mathbb{S}\otimes\mathbb{C}^2} = \pm \overline{h}_{\mathbb{S}\otimes\mathbb{C}^2} \tag{3.27}$$

(with the same sign $\pm$ as above). It follows that the restriction of $h_{\mathbb{S}\otimes\mathbb{C}^2}$ to symplectic Majorana spinors is a real (or purely imaginary) bilinear form invariant under the group $\mathrm{Spin}_0(t,s)\otimes\mathrm{SU}(2)$, which is the subgroup of $\mathrm{Spin}_0(t,s)\otimes\mathrm{U}(2)$, which preserves the subspace of symplectic Majorana spinors. Note that $\mathrm{SU}(2) = \mathrm{U}(2)\cap\mathrm{GL}(1,\mathbb{H}) = \mathrm{Sp}(2,\mathbb{C})\cap\mathrm{GL}(1,\mathbb{H})$ is precisely the centralizer of $\mathrm{Spin}_0(t,s)$ in the group of automorphisms of that real bilinear form on $\mathbb{S}_{SM}$. The centralizer of $\mathrm{Spin}_0(t,s)$ in the larger group of automorphisms of the sesquilinear form on $\mathbb{S}\otimes\mathbb{C}^2$ is $\mathrm{U}(2) = \mathrm{U}(1)\times\mathrm{SU}(2)$, the automorphism group of the standard hermitean form $h_{\mathbb{C}^2}$ on $\mathbb{C}^2$.

Note that we have also

$$\mathrm{SU}(2) = \mathrm{Sp}(2,\mathbb{C}) \cap \mathrm{U}(2) = \mathrm{USp}(2)\,. \tag{3.28}$$

Therefore the internal indices i,j are often called USp(2) indices.

Admissible bilinear forms and super-Poincaré algebras

Above we discussed the notion of an admissible bilinear form on the real spinor module S of $\mathrm{Spin}(t,s)$. We recall that these are bilinear forms β, which have a definite symmetry $\sigma(\beta) = \pm 1$ and type $\tau(\beta) = \pm 1$, and that such forms are automatically $\mathrm{Spin}_0(t,s)$ invariant. The classification of all super-Poincaré algebras for a given dimension (t,s) is equivalent to finding all admissible bilinear forms β, which satisfy the additional condition $\sigma(\beta)\tau(\beta) = +1$ [41]. In fact, to classify super-Poincaré algebras, one needs to classify the possible Lie brackets $\Gamma : S \times S \to V$ which give $\mathrm{so}(V) + V + S$ the structure of a super-Poincaré algebra.[17] Obviously Γ has to be symmetric, because S is in the odd part, and the Jacobi identity is equivalent to the $\mathrm{Spin}_0(t,s)$-equivariance of Γ, since V acts trivially on S. Any admissible bilinear form β defines such a Lie bracket Γ_β on $\mathrm{so}(V) + V + S$ by

$$\langle e^\mu, \Gamma_\beta(\lambda,\chi)\rangle := \beta(\gamma^\mu\lambda,\chi)\,. \tag{3.29}$$

Here $\lambda,\chi \in S$ and $\langle\cdot,\cdot\rangle$ is the natural pairing $V^*\times V \to \mathbb{R}$, $\langle e^\mu, e_\nu\rangle = \delta^\mu_\nu$ for the dual basis (e_ν) of V. Since β is admissible, the vector-valued bilinear form Γ_β is $\mathrm{Spin}_0(t,s)$-equivariant. Moreover the condition $\sigma(\beta)\tau(\beta) = +1$ implies that the Lie bracket $S\times S \to V$ is symmetric, so that on obtains a super Lie algebra. Conversely, it has been shown [41] that any super-Poincaré algebra with odd part S is defined by a linear combination of such maps Γ_β.

[17] In other words one needs to classify the consistent anticommutation relations between the supercharges.

3.2 Minimal supersymmetry in 5 dimensions

In this section we specialize to dimension (1,4) and construct the minimal supersymmetry algebra in terms of Dirac spinors and symplectic Majorana spinors. We also prove that the R-symmetry group is SU(2).

The admissible bilinear forms on the spinor module of $\mathrm{Spin}(1,4)$

So far our discussion applied to arbitrary (t,s). We now specialize to the case of 5-dimensional Minkowski space, $(t,s)=(1,4)$. Since the total dimension $n=t+s=5$ is odd, the complex vector space of $\mathrm{Spin}_0(1,4)$-invariant complex bilinear forms on the complex spinor module $\mathbb{S}$ is one-dimensional. This means that, up to scale, there is only one non-zero invariant bilinear form C. For $n=5$ the symmetry and type of this form are given by $\sigma(C)=-1$ and $\tau(C)=+1$ [41, 40]. Since $n=t+s=5$ is odd and $t-s\equiv 5 \pmod 8$, the complex spinor module $\mathbb{S}$ is an irreducible complex $\mathrm{Spin}_0(1,4)$-module of quaternionic type, see [42] table 1. In other words, there are no Weyl spinors, and since $B^*B=-\mathbb{1}$ one cannot impose a Majorana condition. But it turns out to be convenient to rewrite Dirac spinors as symplectic Majorana spinors, i.e., as pairs of Dirac spinors subject to the symplectic Majorana constraint.

We remark that for $(t,s)=(1,4)$ the real spinor module S is irreducible of quaternionic type and coincides with the complex spinor module $\mathbb{S}$, see [42] table 1. The real vector space $(S^*\otimes S^*)^{\mathrm{Spin}_0(1,4)}$ of $\mathrm{Spin}_0(1,4)$-invariant bilinear forms on the real spinor module S is four-dimensional and admits a basis $(\beta_0,\beta_1,\beta_2,\beta_3)$ consisting of non-degenerate admissible forms with the following invariants, see [41]:

i	0	1	2	3
$(\sigma(\beta_i),\tau(\beta_i))$	$(+1,-1)$	$(-1,-1)$	$(-1,+1)$	$(-1,+1)$

This list shows that only one of the admissible forms β_i, namely β_1, satisfies $\sigma(\beta)\tau(\beta)=+1$ and gives hence rise to a super-Poincaré algebra, according to the algorithm of [41]. In other words, the vector space $(\vee^2 S^*\otimes V)^{\mathrm{Spin}_0(1,4)}$ of super-Poincaré algebra structures with odd part S is one-dimensional and is generated by Γ_{β_1}:

$$(\vee^2 S^*\otimes V)^{\mathrm{Spin}_0(1,4)}=\mathbb{R}\Gamma_{\beta_1}\,. \tag{3.30}$$

Here $\vee^2 S^*$ denotes the symmetric tensor square of the $\mathrm{Spin}(t,s)$-module S^*.

It is easy to check that, up to scale,

$$\beta_0=\operatorname{Im} h_{\mathbb{S}}\,,\qquad \beta_1=\operatorname{Re} h_{\mathbb{S}}\,,\qquad \beta_2=\operatorname{Im} C\,,\qquad \beta_3=\operatorname{Re} C\,, \tag{3.31}$$

where we are using the fact that $\mathbb{S}=S$ for $(t,s)=(1,4)$. The complex bilinear form C and the sesquilinear form $h_{\mathbb{S}}$ on $\mathbb{S}$ were introduced in section 3.1.

The admissible bilinear forms in terms of symplectic Majorana spinors

Next we describe the forms β_i in terms of symplectic Majorana spinors $\mathbb{S}_{SM}\subset\mathbb{S}\otimes\mathbb{C}^2$. We can consider the four real valued bilinear forms $\operatorname{Re} b$, $\operatorname{Im} b$, $\operatorname{Re} h_{\mathbb{S}\otimes\mathbb{C}^2}$ and $\operatorname{Im} h_{\mathbb{S}\otimes\mathbb{C}^2}$ on the

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space of double spinors. The complex bilinear form b and the sesquilinear form $h_{\mathbb{S}\otimes\mathbb{C}^2}$ on $\mathbb{S}\otimes\mathbb{C}^2$ were introduced in section 3.1. For $(t,s)=(1,4)$ the bilinear form b is symmetric and the sesquilinear form $h_{\mathbb{S}\otimes\mathbb{C}^2}$ is anti-hermitean. This shows that

$$\sigma(\mathrm{Re}\, b)=+1\,,\qquad \sigma(\mathrm{Im}\, b)=+1\,,\qquad \sigma(\mathrm{Re}\, h_{\mathbb{S}\otimes\mathbb{C}^2})=-1\,,\qquad \sigma(\mathrm{Im}\, h_{\mathbb{S}\otimes\mathbb{C}^2})=+1\,. \quad (3.32)$$

Moreover, the restriction b_{SM} of ib (or of $ih_{\mathbb{S}\otimes\mathbb{C}^2}$) to the subspace $\mathbb{S}_{SM}\subset\mathbb{S}\otimes\mathbb{C}^2$ is a non-degenerate real-valued bilinear form on $\mathbb{S}_{SM}$. In particular, $\mathrm{Re}\, b=\mathrm{Re}\, h_{\mathbb{S}\otimes\mathbb{C}^2}$ vanish on $\mathbb{S}_{SM}$. Notice that $\sigma(b_{SM})=+1$. Since, up to scale, there is only one $\mathrm{Spin}_0(1,4)$-invariant real symmetric bilinear form on $S=\mathbb{S}_{SM}$, we conclude that b_{SM} is proportional to β_0. Notice that although $\mathbb{S}_{SM}\subset\mathbb{S}\otimes\mathbb{C}^2=\mathbb{S}\oplus\mathbb{S}$ is *not* a Clifford submodule it carries nevertheless an intrinsic Clifford module structure due to the (real) isomorphism $\mathbb{S}\cong\mathbb{S}_{SM}$,

$$\mathbb{S}\ni\lambda\mapsto\lambda_{SM}:=\lambda\oplus j_{\mathbb{S}}\lambda\in\mathbb{S}_{SM}\subset\mathbb{S}\oplus\mathbb{S}\,. \quad (3.33)$$

The intrinsic Clifford multiplication γ^μ_{SM} is given by

$$\gamma^\mu_{SM}\lambda_{SM}=(\gamma^\mu\lambda)_{SM}=-i\gamma^\mu(i\lambda)_{SM}\neq\gamma^\mu\lambda_{SM}\,, \quad (3.34)$$

cf. (3.16). To obtain the remaining admissible bilinear forms β_1, β_2 and β_3 it suffices to consider the forms $b_{SM}(i_{SM}\cdot,\cdot)$, $b_{SM}(j_{SM}\cdot,\cdot)$ and $b_{SM}(k_{SM}\cdot,\cdot)$, respectively, where i_{SM} is the intrinsic $\mathrm{Pin}(1,4)$-invariant complex structure, j_{SM} is the intrinsic $\mathrm{Spin}(1,4)$-invariant quaternionic structure given by

$$i_{SM}\lambda_{SM}=(i_{\mathbb{S}}\lambda)_{SM}\,, \quad (3.35)$$
$$j_{SM}\lambda_{SM}=(j_{\mathbb{S}}\lambda)_{SM}\,, \quad (3.36)$$

and $k_{SM}=i_{SM}j_{SM}$.

The Clifford algebra $Cl_{1,4}$

It follows from the classification of Clifford algebras that $Cl_{1,4}\cong\mathbb{C}(4)$. We shall now describe an explicit isomorphism, from which we will easily recover the above results concerning the spinor module S of $\mathrm{Spin}(1,4)$. We consider the real vector space $\mathbb{H}^2$ and denote by $L_q:\mathbb{H}^2\to\mathbb{H}^2$ the left-multiplication and by $R_q:\mathbb{H}^2\to\mathbb{H}^2$ the right-multiplication by a quaternion $q\in\mathbb{H}$. We put $I:=R_i$, $J:=R_j$ and $K:=IJ=-R_k$. We define the following operators on $\mathbb{H}^2$:

$$\gamma^0:=ID\,,\qquad\gamma^1:=IEL_i\,,\qquad\gamma^2:=IEL_j\,,\qquad\gamma^3:=IEL_k\,,\qquad\gamma^4:=IED\,, \quad (3.37)$$

where

$$E\begin{pmatrix}q_1\\q_2\end{pmatrix}:=\begin{pmatrix}q_1\\-q_2\end{pmatrix},\qquad D\begin{pmatrix}q_1\\q_2\end{pmatrix}:=\begin{pmatrix}q_2\\q_1\end{pmatrix}. \quad (3.38)$$

One can immediately check that the γ^μ satisfy the Clifford relation for the 5-dimensional Minkowski metric η and hence define on $\mathbb{H}^2$ the structure of a $Cl_{1,4}$-module. By dimensional reasons this module is irreducible and provides a faithful representation of the Clifford algebra. The $Cl_{1,4}$-invariant complex structure I provides the identifications $\mathbb{H}^2 = \mathbb{C}^4$ and $Cl_{1,4} = \mathbb{C}(4)$, where $\mathbb{K}(l)$ denotes the full matrix algebra of rank l over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. The even Clifford algebra $Cl^0_{1,4} \subset Cl_{1,4}$ generated by the products $\gamma^\mu\gamma^\nu$ corresponds to $\mathbb{H}(2) \subset \mathbb{C}(4)$, the centralizer of the quaternionic structure J in $Cl_{1,4} = \mathbb{C}(4)$. Now we can identify the spinor module $S = \mathbb{S}$ with $\mathbb{H}^2 = \mathbb{C}^4$ and the connected spinor group $\mathrm{Spin}_0(1,4)$ with $\mathrm{USp}(2,2) = \mathrm{Sp}(4,\mathbb{C}) \cap \mathrm{U}(2,2)$, where the complex symplectic structure corresponds to the bilinear form C introduced in 3.1 and the indefinite hermitean metric corresponds to the sesquilinear form $ih_{\mathbb{S}}$ (note that $h_{\mathbb{S}}$ is anti-hermitean in the minkowskian case $t = 1$). The latter is the complex part of the standard indefinite quaternionic-hermitean metric on $\mathbb{H}^2 = \mathbb{H}^{1,1} = \mathbb{C}^{2,2} = \mathbb{R}^{4,4}$:

$$(q, q') \mapsto \overline{q}_1 q'_1 - \overline{q}_2 q'_2 \,, \tag{3.39}$$

where now $q = (q_i)$, $q' = (q'_i)$, $i = 1, 2$. The real part of that form is the scalar product $g = \beta_0$ of signature $(4,4)$, the i-imaginary part is $\beta_1 = g(I\cdot,\cdot)$ the j-imaginary part is $\beta_2 = g(J\cdot,\cdot)$ and the k-imaginary part is $\beta_3 = g(K\cdot,\cdot)$. The symmetry and type of these forms, indicated above, can be easily checked using this model.

The 5-dimensional supersymmetry algebra

According to [41], the minimal supersymmetry algebra in $(1,4)$-dimensions is given by Γ_{β_1}, where β_1 is the unique (up to scale) admissible real bilinear form on the spinor module S. In the model $S = \mathbb{H}^2$, described above, it is given by the symplectic form $\beta_1 = g(I\cdot,\cdot)$. Under the identifications $S = \mathbb{S}$ and $S \cong \mathbb{S}_{SM}$ it corresponds to the forms $\mathrm{Re}\, h_{\mathbb{S}}$ and $b_{SM}(i_{SM}\cdot,\cdot)$. In standard physics notation, the corresponding supersymmetry algebra with Dirac spinors as supercharges is given by:

$$\{Q_\alpha, Q_\beta\} = \mathrm{Re}((\gamma^\mu A^{-1})_{\alpha\beta}) P_\mu \,, \tag{3.40}$$

where $A^{-1} = (A_{\alpha\beta})$ is the inverse of $A = (A^{\alpha\beta})$. Thus we have four independent complex (or eight independent real) supertransformations. This is the smallest supersymmetry algebra in 5 dimensions.

We now move on to the standard form of the 5-dimensional supersymmetry algebra, which is [43]:

$$\{Q_{i\alpha}, Q_{j\beta}\} = -\frac{1}{2}\epsilon_{ij}(\gamma^\mu \mathsf{C}^{-1})_{\alpha\beta} P_\mu \,. \tag{3.41}$$

Here $\mathsf{C}^{-1} = (\mathsf{C}_{\alpha\beta})$ is the inverse of the charge conjugation matrix and the supercharges are subject to the symplectic Majorana condition (3.17).

Before imposing this constraint, the algebra (3.41) corresponds, up to a factor, to the $\mathrm{Spin}_0(1,4)$-invariant bracket Γ_b associated, by the universal formula (3.29), to the admissible complex bilinear form $b = C \otimes \epsilon$ on the space of double spinors $\mathbb{S} \otimes \mathbb{C}^2$. The supersymmetry charges are, hence, pairs of Dirac spinors and we have eight independent complex supertransformations. This is not the smallest supersymmetry algebra in 5 dimensions.

But when imposing the symplectic Majorana condition (3.17) on the supercharges $Q_{i\alpha}$, we are left with eight real or four complex supercharges and the algebra (3.41) becomes isomorphic to (3.40). We emphasize that the restriction of the bracket Γ_b to the space

$\mathbb{S}_{SM} \subset \mathbb{S} \otimes \mathbb{C}^2$ is real-valued, although the restriction of the bilinear form b is purely imaginary. The reason is that the definition of the bracket Γ_b involves the Clifford multiplication γ^μ, which anticommutes with the real structure ρ due to (3.16). More explicitly, we can use the isomorphism $\mathbb{S}_{SM} \ni (\lambda^1, \lambda^2)^T = (\lambda^1, j_{\mathbb{S}}(\lambda^1))^T \to \lambda^1 \in \mathbb{S}$ to find $\Gamma_b|_{SM} = -2\Gamma_{\beta_1}$, which maps (3.41) to (3.40).

The R-symmetry group

Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 = (\mathrm{so}(V) + V) + S$ be a super-Poincaré algebra, $V = \mathbb{R}^{t,s}$. According to [41] the Lie superbracket is a $\mathrm{Spin}_0(t,s)$-equivariant symmetric bilinear map of the form $\Gamma_\beta : S \times S \to V$, see (3.29), where β is a linear combination of admissible bilinear forms. The *R-symmetry group* of $\mathfrak{g}$ is the group

$$G_R := \{\phi \in \mathrm{Aut}(\mathfrak{g}) | \, \phi|_{\mathfrak{g}_0} = \mathrm{Id}\} \tag{3.42}$$

of automorphisms of $\mathfrak{g}$ which act trivially on the even part. Notice that G_R can be considered as a subgroup of $\mathrm{GL}(S)$, since the action on $\mathfrak{g}_0$ is trivial. More precisely,

$$G_R = G_R(\mathfrak{g}) = \{\phi \in \mathrm{GL}(S) | \, [\phi, \mathrm{so}(V)] = 0 \quad \text{and} \quad \Gamma_\beta(\phi\lambda, \phi\chi) = \Gamma_\beta(\lambda, \chi) \quad \text{for all } \lambda, \chi \in S\}\,, \tag{3.43}$$

which is the centralizer of $\mathrm{Spin}_0(t,s)$ in the automorphism group of the vector-valued bilinear form Γ_β.

Next we describe the R-symmetry group in the case $V = \mathbb{R}^{t,s} = \mathbb{R}^{1,4}$ in terms of symplectic Majorana spinors $S \cong \mathbb{S}_{SM} \subset \mathbb{S} \otimes \mathbb{C}^2$. This formalism has the advantage that the R-symmetry group acts on the internal space $\mathbb{C}^2$. Here G_R is identified with a subgroup of the complex Lie group $\mathrm{GL}(\Sigma)$, $\Sigma = \mathbb{S} \otimes \mathbb{C}^2$:

$$G_R = \{\phi \in \mathrm{GL}(\Sigma) | [\phi, \mathrm{so}(V)] = 0\,, \quad \phi\rho = \rho\phi \quad \text{and} \quad \Gamma_b(\phi\lambda, \phi\chi) = \Gamma_b(\lambda, \chi) \quad \text{for all } \lambda, \chi \in \Sigma\}\,. \tag{3.44}$$

Here $b = C \otimes \epsilon$ is the symmetric bilinear form on the the space of double spinors Σ and ρ is the real structure which defines the symplectic Majorana spinors. To determine this group, let us first remark that by Schur's Lemma the first condition $[\phi, \mathrm{so}(V)] = 0$ implies that $\phi = \mathrm{Id} \otimes \varphi$, $\varphi \in \mathrm{GL}(2, \mathbb{C})$. The reality condition $\phi\rho = \rho\phi$ is equivalent to $\varphi j_{\mathbb{C}^2} = j_{\mathbb{C}^2}\varphi$, i.e., to $\varphi \in \mathrm{GL}(1, \mathbb{H}) \subset \mathrm{GL}(2, \mathbb{C})$. An element $\phi = \mathrm{Id} \otimes \varphi$ satisfies the third condition $\Gamma_b(\phi\cdot, \phi\cdot) = \Gamma_b$ if and only if $\varphi \in \mathrm{Sp}(2, \mathbb{C})$. Thus we have proven that the R-symmetry group of (3.41) subject to the reality constraint (3.17) is given by

$$G_R = \mathrm{Id} \otimes (\mathrm{Sp}(2, \mathbb{C}) \cap \mathrm{GL}(1, \mathbb{H})) = \mathrm{Id} \otimes \mathrm{SU}(2) \cong \mathrm{SU}(2)\,. \tag{3.45}$$

3.3 4-dimensional $\mathcal{N} = 2$ supersymmetry from dimensional reduction

Next we discuss the relation of the $(1,4)$-dimensional supersymmetry algebra to the $\mathcal{N} = 2$ supersymmetry algebras in space-times of dimensions $(1,3)$ and $(0,4)$. Starting from 5 dimensions, these supersymmetry algebras are obtained by dimensional reduction over a space-like or time-like direction, respectively. We perform a standard Kaluza-Klein reduction, i.e., we take all dependencies on the reduced direction to be trivial. At the level of the algebra this amounts to setting to zero one space-like or time-like momentum operator (= translation operator).

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It is convenient to take the 5-dimensional space-time indices to be $\mu, \nu = 0, 1, 2, 3, 5$. The corresponding gamma-matrices are

$$\gamma^0,\, \gamma^1,\, \gamma^2,\, \gamma^3,\, \gamma^5\,, \tag{3.46}$$

where, as already mentioned, γ^0 is anti-hermitean whereas the other matrices are hermitean. We use a representation where these matrices are related by [40]:

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3\,, \qquad \gamma^5\gamma^5 = \mathbb{1}\,. \tag{3.47}$$

Mathematically, this corresponds to the fact that the Clifford algebra $\mathbb{C}l_5 = \mathbb{C}(4) \oplus \mathbb{C}(4)$ has two irreducible modules, which differ by the value of $i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = \pm\mathbb{1}$.

Upon dimensional reduction, one of the five Lorentz indices becomes an internal index, which we denote by $*$. When reducing over a space-like dimension we take $* = 5$, in reduction over time we have $* = 0$. The Clifford algebras are generated by

$$\begin{aligned} &\gamma^0,\, \gamma^1,\, \gamma^2,\, \gamma^3\,, && \text{for } Cl_{1,3} \text{ and} \\ &\gamma^1,\, \gamma^2,\, \gamma^3,\, \gamma^5\,, && \text{for } Cl_{0,4}\,. \end{aligned} \tag{3.48}$$

The additional generator γ^* can be used to impose a chirality constraint. If the number of space-time dimensions is even, the spinor module decomposes into two irreducible eigenspaces with respect to the so-called volume element [61], which is proportional to the product of all Clifford generators. In physical terminology this is the 'generalized γ^5-matrix.'[18] The minimal spinor representation of 5 dimensions becomes reducible in 4 dimensions, as Dirac spinors decompose into Weyl spinors.

In the $(1, 3)$ theory $\gamma^* = \gamma^5$ is the standard chirality matrix. For later notational convenience we set $\Gamma_* := \gamma^5$. The corresponding projector can be used to decompose spinors:

$$\lambda^i = \lambda^i_+ + \lambda^i_-\,, \qquad \text{where } \lambda^i_\pm := \Gamma_\pm\lambda^i = \frac{1}{2}(\mathbb{1} \pm \Gamma_*)\lambda^i\,. \tag{3.49}$$

The 5-dimensional symplectic Majorana constraint (3.17) can be written as

$$(\lambda^i)^* = \mathsf{C}\gamma_0\epsilon_{ij}\lambda^j\,, \tag{3.50}$$

where we used $A = \gamma_0$. Chiral decomposition of the symplectic Majorana constraint gives

$$(\lambda^i_\pm)^* = \mathsf{C}\,\gamma_0\,\epsilon_{ij}\,\lambda^j_\mp\,. \tag{3.51}$$

Note that the chiral projection is not compatible with the symplectic Majorana condition, which is not surprising, since there are no symplectic Majorana-Weyl spinors for $(t, s) = (1, 3)$. If one wants to work with irreducible spinors, one can reformulate the theory either in terms of Weyl spinors, or, equivalently, in terms of (standard, not symplectic) Majorana spinors. For our purposes, however, it is more convenient to stick to symplectic Majorana

[18] Taken literally, chirality refers to the distinction between left-handed and right-handed frames in odd-dimensional space. But in a slight abuse of language we will generally refer to the above decomposition as 'chiral.'

spinors, because they exist in all three space-time signatures $(1,4)$, $(1,3)$ and $(0,4)$. Since a considerable part of the literature uses Majorana spinors, we briefly review the relation between standard and symplectic Majorana spinors in $(1,3)$ dimensions in the appendix (see in particular (A.13)).

Let us now consider the chiral decomposition for dimension $(0,4)$. Now γ^0 can be used to impose a chirality constraint. Since γ^0 squares to $-\mathbb{1}$ rather than $\mathbb{1}$, we take the chirality matrix to be[19]

$$\Gamma_*^{\mathrm{E}} := -i\gamma^0\,, \tag{3.52}$$

with corresponding projector

$$\Gamma_\pm^{\mathrm{E}} := \frac{1}{2}(\mathbb{1} \pm \Gamma_*^{\mathrm{E}})\,. \tag{3.53}$$

Then spinors decompose according to

$$\lambda^i = \xi_+^i + \xi_-^i\,, \qquad \xi_\pm^i := \Gamma_\pm^{\mathrm{E}}\lambda^i\,. \tag{3.54}$$

Using $B = C\gamma_0$, the symplectic Majorana constraint (3.17) becomes

$$\left(\lambda^i\right)^* = -\,i\,\mathsf{C}\,\Gamma_*^{\mathrm{E}}\,\epsilon_{ij}\,\lambda^j\,. \tag{3.55}$$

Using the chiral projection we find

$$(\xi_\pm^i)^* = -\,i\,\mathsf{C}\,\Gamma_*^{\mathrm{E}}\,\epsilon_{ij}\,\xi_\pm^j\,, \tag{3.56}$$

which shows that in dimension $(0,4)$ the chiral and symplectic Majorana constraints are compatible, i.e., there are symplectic Majorana-Weyl spinors. This is in accordance with the general analysis, see [40, 42].

We now turn to the supersymmetry algebra. Since the supercharges in all three cases are symplectic Majorana spinors, the reduction is straightforward:

$$\{Q_{i\alpha}, Q_{j\beta}\} = -\frac{1}{2}\epsilon_{ij}\left(\gamma^\mu\mathsf{C}^{-1}\right)_{\alpha\beta}P_\mu = -\frac{1}{2}\epsilon_{ij}\left(\gamma^m\mathsf{C}^{-1}\right)_{\alpha\beta}P_m - \frac{1}{2}\epsilon_{ij}\left(\gamma^*\mathsf{C}^{-1}\right)_{\alpha\beta}P_*\,. \tag{3.57}$$

Since $P_* f = i\frac{\partial}{\partial x^*}f = 0$ for fields f which do not depend on the internal direction x^*, the resulting supersymmetry algebra in 4 dimension is

$$\{Q_{i\alpha}, Q_{j\beta}\} = -\frac{1}{2}\epsilon_{ij}\left(\gamma^m\mathsf{C}^{-1}\right)_{\alpha\beta}P_m\,. \tag{3.58}$$

More generally, P_* could act as a real central charge on the Fourier modes of the 5-dimensional fields. But we discard all non-constant Fourier modes, so that P_* acts trivially.[20]

[19]The choice of the overall sign is suggested by the dimensional reduction of the 5-dimensional supersymmetry transformations. See section 5.2.

[20]We have also omitted the real central charge of the $(1,4)$-dimensional supersymmetry algebra, because we are only interested in massless states, on which it acts trivially. If one includes the 5-dimensional central charge in dimensional reduction and keeps P_*, they combine into the complex central charge of the 4-dimensional $\mathcal{N}=2$ supersymmetry algebra. Again, this central charge acts trivially on massless states and is irrelevant for our purpose.

Notice that the super Lie algebra (3.58), subject to the symplectic Majorana constraint, is the standard $\mathcal{N} = 2$ super-Poincaré algebra in dimension $(1,3)$. Using the formulae given in the appendix one can rewrite the supercharges in terms of two Majorana spinors. This algebra is not the minimal one, which is generated by just one Majorana spinor, or, equivalently, one Weyl spinor of supercharges [62].

In dimension $(0,4)$ we also get a real $\mathcal{N} = 2$ super-Poincaré algebra with odd part $\mathbb{S} = \mathbb{S}_+ + \mathbb{S}_-$ and $\{\mathbb{S}_+, \mathbb{S}_+\} = \{\mathbb{S}_-, \mathbb{S}_-\} = 0$. In this case we do not need to distinguish between the real and complex spinorial modules: $S = \mathbb{S}$ and $\mathbb{S}_\pm = S_\pm$. In euclidean 4-space the above super Lie algebra is a minimal super-Poincaré algebra, in contrast to the Minkowski case. This is clear from the fact that only the superbrackets between $\mathbb{S}_+$ and $\mathbb{S}_-$ are non-vanishing. Moreover, although the vector space $(\vee^2 S^* \otimes V)^{\mathrm{Spin}(0,4)}$ of super-Poincaré algebra structures with odd part S is 4-dimensional [41], one can show that any other nontrivial $\mathcal{N} = 2$ super-Poincaré algebra is isomorphic to (3.58) subject to the symplectic Majorana constraint.

Obviously, the two 4-dimensional supersymmetry algebras inherit the R-symmetry group SU(2) of the 5-dimensional algebra. But since γ^* does not represent any element of the dimensionally reduced Clifford algebra ($Cl_{1,3}$ or $Cl_{0,4}$), it now generates an internal transformation. This is a candidate for a new R-symmetry, because it commutes with the even part $Cl^0_{t,s}$ of $Cl_{t,s}$, for both $(t,s) = (1,3), (0,4)$. Since the invariant bilinear form C on the spinor module of $\mathrm{Spin}(5,\mathbb{C})$ has type $\tau(C) = +1$ and γ^* anticommutes with all other gamma matrices, the transformation

$$Q_i \to \exp(\gamma^* \Phi) Q_i \tag{3.59}$$

leaves (3.58) invariant for arbitrary complex transformation parameters Φ. We still have to check the reality constraint

$$(\exp(\gamma^* \Phi) Q^i)^* = \mathsf{C} \gamma_0 \epsilon_{ij} \exp(\gamma^* \Phi) Q^j \,. \tag{3.60}$$

In dimension $(1,3)$ we have $\gamma^* = \gamma^5$, which is hermitean and anticommutes with γ^0. Thus we find

$$\exp((\gamma^5)^* \Phi^*) = \exp(-(\gamma^5)^T \Phi) \,. \tag{3.61}$$

Using again that γ^5 is hermitean we learn that Φ must be imaginary. Thus the new R-symmetry is

$$Q_i \to \exp(i\gamma^5 \phi) Q_i \,, \qquad \phi \in \mathbb{R} \,. \tag{3.62}$$

The group generated by γ^5 is isomorphic to U(1) and acts chirally, i.e., the chiral components $Q_{i\,\pm}$ transform with opposite phases $e^{\pm i\phi}$. Thus we find the well-known R-symmetry group of $\mathcal{N} = 2$ supersymmetry [62]:

$$G_R = \mathrm{U}(2) \simeq \mathrm{U}(1) \times \mathrm{SU}(2) \simeq \mathrm{U}(1) \times \mathrm{USp}(2) \,. \tag{3.63}$$

In dimension $(0, 4)$ we have $\gamma^* = \gamma^0$, which is anti-hermitean and commutes with γ^0. We find

$$\exp((\gamma^0)^* \Phi^*) = \exp((\gamma^0)^T \Phi) , \tag{3.64}$$

and using that γ^0 is anti-hermitean we see that Φ must be imaginary, $\Phi = -i\phi$, with $\phi \in \mathbb{R}$. Since $(\gamma^0)^2 = -\mathbb{1}$ we get a non-compact R-symmetry group. The R-symmetry transformation is

$$Q_i \to \exp(-i\gamma^0 \phi) Q_i = \exp(\Gamma_*^{\mathrm{E}} \phi) Q_i , \tag{3.65}$$

with real ϕ. Since $(\Gamma_*^{\mathrm{E}})^2 = \mathbb{1}$, the generator Γ_*^{E} has eigenvalues ± 1 (rather than $\pm i$) and acts by chiral scale transformations:

$$Q_{i\,\pm} \to e^{\pm \phi} Q_{i\,\pm} . \tag{3.66}$$

Taking into account the obvious symmetry $Q_{i\,\pm} \to -Q_{i\,\pm}$ of (3.58), we find that the R-symmetry group contains an additional subgroup $\mathrm{SO}(1, 1)$, which commutes with $\mathrm{SU}(2)$. Analyzing the spinor representation in dimension $(0, 4)$, with the same methods that were used above for dimensions $(1, 4)$, one can prove that the full R-symmetry group of 4-dimensional euclidean $\mathcal{N} = 2$ supersymmetry is

$$G_R = \mathrm{SO}(1, 1) \times \mathrm{SU}(2) . \tag{3.67}$$

The fact that the abelian factor of the $\mathcal{N} = 2$ R-symmetry becomes non-compact for euclidean signature has been observed by various authors, starting from [20]. In [17], which studies the dimensional reduction from dimension $(1, 5)$ to dimension $(0, 4)$, the $\mathrm{SO}(1, 1)$ factor was related to the internal part of the 6-dimensional Lorentz group. Note that this is different for our reduction, which starts in dimension $(1, 4)$, because there is no subgroup $\mathrm{SO}(1, 1)$ of $\mathrm{SO}(1, 4)$ which commutes with $\mathrm{SO}(4)$. Instead, we found that the new part of the R-symmetry group arises from the Clifford algebra.

3.4 Commuting versus anticommuting spinors

In supersymmetric field theory one uses spinor fields with components which are not real or complex numbers, but 'Grassmann numbers', i.e., elements of a Grassmann algebra defined by a system of anticommuting generators. In geometrical terms, this means to work with super vector spaces and super manifolds [63, 64, 65].

Let us explain what this means for the cases we are interested in. If S is the real spinor module underlying the supersymmetry algebra of our field theory, then we replace it by ΠS, which is the spinor module considered as a purely odd super vector space of dimension $(0\,|\,m)$, where m is the dimension of S. (Π is called the parity change functor in [65].) The elements of ΠS are called anticommuting spinors. Since we want to consider spinor fields, we also need to identify the appropriate spinor bundle. For commuting spinors the spinor bundle is $S(\mathbb{R}^{t,s}) = \mathbb{R}^{t,s} \times S \to \mathbb{R}^{t,s}$, the trivial bundle over space-time $\mathbb{R}^{t,s}$ with fibre S. It is trivial since we only consider flat simply connected space-times. To define anticommuting spinor fields, it is not sufficient to replace the fibre S by ΠS, as the resulting super vector bundle $\mathbb{R}^{t,s} \times \Pi S \to \mathbb{R}^{t,s}$ of rank $(0|m)$ has no sections other than the zero section. The reason simply is that the m local components of a section must be odd superfunctions, which can be non-zero only if the base of the bundle is a supermanifold with a non-trivial odd part. Therefore one replaces the space-time $\mathbb{R}^{t,s}$ by the flat superspace $\mathbb{R}^{t,s|m} = \mathbb{R}^{t,s} \times N$, where N is an internal, purely odd parameter space of dimension m. The super vector bundle $\Pi S(\mathbb{R}^{t,s|m}) := \mathbb{R}^{t,s|m} \times \Pi S \to \mathbb{R}^{t,s|m}$ has non-trivial sections. An anticommuting spinor field is, by definition, a section of the bundle $\Pi S(\mathbb{R}^{t,s|m})$ This is used in the field theoretic part of the paper. The above construction can be easily generalized to the case of space-times with non-trivial spinor bundle.

Note that going from commuting to anticommuting spinor components changes the symmetry properties of the bilinear forms in the obvious way. This should be kept in mind, because the actions and supersymmetry transformation rules appearing in the following sections involve spinor bilinears, which are built out of anticommuting spinors. In contrast, we used commuting spinors in this section.

4. Vector multiplets in 5 dimensions

The aim of this section is to construct the $\mathcal{N} = 2$ rigidly supersymmetric lagrangian[21] for abelian vector multiplets in dimension (1,4). Our motivation is that we need it as the starting point for dimensional reduction, in order to explore the properties of the resulting lagrangians in dimensions $(1,3)$ and $(0,4)$.

5-dimensional supersymmetric field theories have been studied using string theory in [44] and, subsequently, in other papers including [50, 51]. For abelian vector multiplets the lagrangian is completely determined by a real cubic polynomial, the prepotential [44]. Since none of the above references specifies the explicit lagrangian and supersymmetry rules, we derive them in the following. The construction makes essential use of the results of [43], where the general lagrangian of 5-dimensional rigid super*conformal* vector multiplets was constructed in the framework of the superconformal calculus. We adapt this construction to the case of *Poincaré* supersymmetry. In both cases one finds that all couplings are encoded in a real prepotential. In the superconformal case this function must be a *homogeneous polynomial of degree three*. Moreover, the superconformal invariance leads to additional terms in the supersymmetry variations. This is due to the fact that the superconformal algebra contains, besides the standard supersymmetry transformation (Q-transformations), a second set of so-called special supersymmetry transformations (S-transformations). Since we are interested in a general super-Poincaré invariant lagrangian, we do not require invariance under scale and special conformal transformations, nor under S-transformations. We therefore have to reanalyze the constraints on the prepotential and we will find that it is now allowed to be an arbitrary *cubic polynomial.* Higher order terms are ruled out by abelian gauge invariance [44].

[21] As explained in the introduction we generally refer to theories with eight real supercharges as $\mathcal{N} = 2$. Other authors call this theory $\mathcal{N} = 1$ supersymmetric, because the underlying supersymmetry algebra is minimal.

The basic building block for our model is the $\mathcal{N}=2$ off-shell vector multiplet [43], which has the following field content (our conventions are summarized in the appendix) :

$$\{A_\mu, \lambda^i, \sigma, Y^{ij}\}. \tag{4.1}$$

Here A_μ is a 5-dimensional one-form, which should be considered as the gauge potential of a connection in a line bundle. As is common in physics, we understand that one-forms and vector fields have been identified using the space-time metric, which explains the name "vector multiplet." The pair λ^i is a symplectic Majorana spinor, and σ is a real scalar field. In order to have the correct number of off-shell degrees of freedom, the multiplet also contains the auxiliary field Y^{ij}, which is a real, symmetric tensor of the R-symmetry group SU(2):

$$Y^{ij} = Y^{ji}\,, \qquad (Y^{ij})^* = Y_{ij} = \epsilon_{ik}\epsilon_{jl}Y^{kl}\,. \tag{4.2}$$

Note that the real structure on symmetric tensors over $\mathbb{C}^2$ used here is simply the tensor square of the SU(2)-invariant quaternionic structure on $\mathbb{C}^2$.

As usual in supersymmetric field theories, the spinors λ are anticommuting (see section 3.4). Note that this changes the symmetry properties of the bilinear forms discussed in the previous section, where we used commuting spinors. The formulae needed for handling expressions containing anticommuting spinors are collected in the appendix.

The kinetic terms for this multiplet are given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\bar{\lambda}\partial\!\!\!/\,\lambda - \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma + Y^{ij}Y_{ij}\,, \tag{4.3}$$

where $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ is the Dirac operator. The action corresponding to this lagrangian is invariant under the following off-shell supersymmetry variations:

$$\begin{aligned}
\delta A_\mu &= \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda\,,\\
\delta Y^{ij} &= -\frac{1}{2}\bar{\epsilon}^{(i}\partial\!\!\!/\lambda^{j)}\,,\\
\delta\lambda^i &= -\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}\epsilon^i - \frac{i}{2}\partial\!\!\!/\sigma\,\epsilon^i - Y^{ij}\epsilon_j\,,\\
\delta\sigma &= \frac{i}{2}\bar{\epsilon}\lambda\,.
\end{aligned} \tag{4.4}$$

Here $\epsilon = (\epsilon^i)$ is the parameter of the supersymmetry transformation, which is an anticommuting symplectic Majorana spinor, and $\bar{\epsilon}$ is its Majorana conjugate, see (3.9). Working in an off-shell formulation has the advantage that the supersymmetry transformations do not depend on equations of motion derived from the lagrangian. Hence they will retain their form when we add further terms to the lagrangian.

We now take N copies of (4.3), and couple them by a symmetric matrix, $a_{IJ}(\sigma)$:[22]

$$\mathcal{L}_{\text{kin}} = \left(-\frac{1}{4}F^I_{\mu\nu}F^{J\,\mu\nu} - \frac{1}{2}\bar{\lambda}^I\partial\!\!\!/\,\lambda^J - \frac{1}{2}\partial_\mu\sigma^I\,\partial^\mu\sigma^J + Y^I_{ij}Y^{J\,ij}\right)a_{IJ}(\sigma)\,. \tag{4.5}$$

[22]One might have expected that in the spinor term the partial derivative is promoted to a covariant derivative with respect to the Levi-Civita connection of a_{IJ}. However, the term containing the connection is identically zero for anticommuting symplectic Majorana spinors.

N is the number of vector multiplets in our model, which are labeled by the indices $I, J \in \{1, \ldots, N\}$. Note that the matrix $a_{IJ}(\sigma)$ is allowed to depend on the scalar fields. In particular, the scalar kinetic term has now been promoted to a non-linear sigma model. The scalar fields σ^I can be interpreted as a map from space-time $\mathbb{R}^{1,4}$ into an N-dimensional riemannian manifold $\mathcal{M}$ with metric $a_{IJ}(\sigma)$.

We require that the variations (4.4) hold for every copy of the vector multiplet separately:

$$
\begin{aligned}
\delta\sigma^I &= \frac{i}{2}\,\bar{\epsilon}\,\lambda^I\,, \\
\delta A^I_\mu &= \frac{1}{2}\,\bar{\epsilon}\,\gamma_\mu\,\lambda^I\,, \\
\delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}\,F^I_{\mu\nu}\,\epsilon^i \;-\; \frac{i}{2}\,\partial\!\!\!/\sigma^I\,\epsilon^i \;-\; Y^{ij\,I}\,\epsilon_j\,, \\
\delta\bar{\lambda}^{iI} &= \frac{1}{4}\,\bar{\epsilon}^i\,\gamma^{\mu\nu}F^I_{\mu\nu} - \frac{i}{2}\,\bar{\epsilon}^i\partial\!\!\!/\,\sigma^I - \bar{\epsilon}_j\,Y^{ij\,I}\,, \\
\delta Y^{ij\,I} &= -\frac{1}{2}\,\bar{\epsilon}^{(i}\partial\!\!\!/\lambda^{j)I}\,.
\end{aligned}
\tag{4.6}
$$

Calculating the supersymmetry variation of the action corresponding to the lagrangian (4.5), we find that the action is invariant up to terms which contain either a derivative $\partial_\mu a_{IJ}(\sigma)$ or a variation $\delta a_{IJ}(\sigma)$ of the metric. These can be combined by rewriting them as $(\frac{\partial}{\partial\sigma^K}\,a_{IJ}(\sigma))\,\partial_\mu\,\sigma^K$ and $(\frac{\partial}{\partial\sigma^K}a_{IJ}(\sigma))\,\delta\sigma^K$, respectively. The non-vanishing terms of the variation are then of the form

$$
\delta S_{\text{kin}} = \int d^5x\,(\ldots)\,\frac{\partial}{\partial\sigma^K}\,a_{IJ}(\sigma)\,. \tag{4.7}
$$

Hence we find that the action arising from (4.5) is supersymmetric if we impose the condition that $a_{IJ}(\sigma)$ is independent of σ.

However, this is not the most general form of the lagrangian if we can add further terms to the action, whose supersymmetry transformations cancel the terms left in (4.7). This can indeed be accomplished by adding interactions of Chern-Simons type, if we require $\frac{\partial}{\partial\sigma^K}\,a_{IJ}(\sigma)$ to be symmetric in all three indices. Then $a_{IJ}(\sigma)$ can be expressed as the second derivative of a function $F(\sigma)$, the prepotential:

$$
a_{IJ}(\sigma) = \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\,F(\sigma)\,. \tag{4.8}
$$

Using the prepotential, we may then rewrite $\frac{\partial}{\partial\sigma^K}\,a_{IJ}(\sigma)$ as

$$
F_{IJK}(\sigma) := \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\,\frac{\partial}{\partial\sigma^K}\,F(\sigma)\,. \tag{4.9}
$$

By construction, $F_{IJK}(\sigma)$ is totally symmetric in all indices.

Having imposed this condition, we now add the following Chern-Simons-like interactions to the lagrangian:

$$
\mathcal{L}_{\text{CS}} = \left(-\frac{1}{24}\,\epsilon^{\mu\nu\lambda\rho\sigma}\,A^I_\mu F^J_{\nu\lambda}F^K_{\rho\sigma} - \frac{i}{8}\,\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K - \frac{i}{2}\bar{\lambda}^{iI}\lambda^{jJ}\,Y^K_{ij}\right)F_{IJK}(\sigma)\,. \tag{4.10}
$$

Calculating the supersymmetry variation of the action corresponding to

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{CS}}\,, \tag{4.11}$$

we find that the action is invariant up to terms which are proportional to the fourth derivative of the prepotential:

$$\delta\left(S_{\text{kin}} + S_{\text{CS}}\right) = \int d^5x\, (\cdots)\, F_{IJKL}(\sigma)\,. \tag{4.12}$$

Here $F_{IJKL}(\sigma) := \frac{\partial}{\partial\sigma^I}\frac{\partial}{\partial\sigma^J}\frac{\partial}{\partial\sigma^K}\frac{\partial}{\partial\sigma^L}F(\sigma)$.

In order to ensure gauge invariance, the prepotential must be restricted to a *polynomial of degree at most 3* and this is sufficient to ensure that the action defined by the lagrangian (4.11) is supersymmetric. The crucial observation is that the Chern-Simons term $\epsilon^{\mu\nu\lambda\rho\sigma}A^I_\mu F^J_{\nu\lambda}F^K_{\rho\sigma}$ is gauge invariant up to partial integration, only. If we allowed for F_{IJK} to be a function of σ, this partial integration would generate a nontrivial term with the partial derivative acting on $F_{IJK}(\sigma)$. But such a term would break the gauge invariance of the action. This forces us to restrict the prepotential F to be a polynomial of at most cubic degree. Since this restriction implies $F_{IJKL}(\sigma) = 0$, the remaining terms in the supersymmetry variation (4.12) vanish identically.

As a result, we arrive at the following general $\mathcal{N} = 2$ vector multiplet lagrangian in dimension $(1,4)$:

$$\begin{aligned}\mathcal{L} = &\left(-\frac{1}{4}F^I_{\mu\nu}F^{J\,\mu\nu} - \frac{1}{2}\bar{\lambda}^I\not{\partial}\lambda^J - \frac{1}{2}\partial_\mu\sigma^I\,\partial^\mu\sigma^J + Y^I_{ij}Y^{J\,ij}\right)a_{IJ}(\sigma) + \\ &+\left(-\frac{1}{24}\epsilon^{\mu\nu\lambda\rho\sigma}A^I_\mu F^J_{\nu\lambda}F^K_{\rho\sigma} - \frac{i}{8}\bar{\lambda}^I\gamma^{\mu\nu}F^J_{\mu\nu}\lambda^K - \frac{i}{2}\bar{\lambda}^{iI}\lambda^{jJ}\,Y^K_{ij}\right)F_{IJK}\,.\end{aligned} \tag{4.13}$$

It is invariant under the supersymmetry transformations given in (4.6) and the prepotential $F(\sigma)$ is restricted to be a polynomial of degree not greater than 3.

The analogous locally supersymmetric action, i.e., the action of N vector multiplets coupled to minimal 5-dimensional supergravity was worked out long ago in [45]. In this case the theory is fully determined by a *homogeneous* polynomial $\mathcal{V}(h^{\hat{I}})$ of degree 3 in $N+1$ variables, $\hat{I} = 0,\ldots,N$. The corresponding scalar manifold $\mathcal{M}_{\text{local}}$ is still N-dimensional, because it is defined by the cubic hypersurface $\mathcal{V}(h^{\hat{I}}) = 1$. This is the defining property of a *projective (or local) very special real manifold* [45, 46, 47]. Above we found the scalar geometry of the corresponding globally supersymmetric theories: the metric $a_{IJ}(\sigma)$ must be the Hessian of a polynomial of degree at most three [44]. This can be taken as the definition of an *affine very special real manifold.* In the case of superconformal theories the cubic function must be homogeneous [16, 43]. If one admits 5-dimensional space-times with non-trivial topology or introduces charged fields, then the coefficients of the cubic polynomial are subject to further constraints [66, 44].

An alternative way to derive (4.13) would be to start with the locally supersymmetric action of [45] and to decouple gravity by sending the Planck mass to infinity. Even without doing so in detail, it is clear that this will give a lagrangian of the form (4.13). In analogy to the rigid limit of 4-dimensional $\mathcal{N} = 2$ vector multiplets (see for example [36]), the rigid limit freezes one variable of the homogeneous cubic polynomial $\mathcal{V}(h^{\hat{I}})$, so that one is left with a general cubic polynomial $F(\sigma^I)$ in the remaining variables. Roughly speaking, the frozen variable corresponds to graviphoton, which is the abelian gauge field in the supergravity multiplet. By inspection of [45], one sees that the terms surviving the rigid limit have the form (4.13).

Our formulation of the theory is not covariant with respect to general coordinate transformations of the scalar manifold $\mathcal{M}$, but only covariant with respect to affine transformations $\sigma^I \to R^I_J \sigma^J + a^I$, with constant, invertible R^I_J, and constant a^I. Thus the scalar fields σ^I are *affine coordinates.* In analogy to $(1,3)$-dimensional $\mathcal{N} = 2$ vector multiplets we will also call them *special coordinates.* There is no principal problem to reformulate the theory in terms of general coordinates, as has been done in (1,3)-dimensional case , see [36, 37]. However, we prefer to work in special coordinates, which are adapted to the symmetries of the theory, since they are the lowest components of $\mathcal{N} = 2$ vector supermultiplets.

For completeness, we give a global (in the mathematical sense) characterization of the scalar manifolds of 5-dimensional rigid vector multiplets: an *affine very special real manifold* is a differentiable manifold equipped with (i) a flat torsion-free connection ∇, and with (ii) a riemannian (or, more generally, a pseudo-riemannian) metric, which, when expressed in local affine coordinates, is the Hessian of a polynomial of degree at most 3. The first condition ensures that the manifold is an affine manifold, i.e., it can be covered with local affine coordinate systems σ^I. In each patch affine coordinates are characterized by $\nabla d\sigma^I = 0$, and coordinates in different patches are related by affine transformations, as above. Therefore the second condition makes sense.

5. Dimensional reduction to 4 dimensions

In this section we perform a standard Kaluza-Klein reduction of the lagrangian (4.13) on a circle S^1, keeping only the massless modes. This corresponds to the limit where the S^1 is shrunk to zero radius, so that all excited Kaluza-Klein states (non-constant Fourier modes) become infinitely heavy and decouple. Taking the compact dimension to be either space- or time-like, we obtain $\mathcal{N} = 2$ supersymmetric lagrangians with minkowskian and euclidean signature, respectively. We then identify the geometric structures underlying these theories and show that they can be mapped to one another.

The section is organized as follows: We first dimensionally reduce the bosonic terms of the lagrangian (4.13) to $(1,3)$ and $(0,4)$ dimensions and discuss the structures of the resulting scalar manifolds. We then determine the supersymmetry variations of the new lagrangians, before we give the complete fermionic terms. Next we show how our results can be generalized to euclidean theories not obtained by dimensional reduction and display the general lagrangian. Finally we explain how the abelian factor of the R-symmetry group is related to the existence of a complex or para-complex structure on the scalar manifold.

5.1 The bosonic sector

5.1.1 Dimensional reduction

We start with the dimensional reduction of the bosonic sector of our 5-dimensional lagrangian (4.13):

$$\mathcal{L}^{(1,4)}_{\text{bos}} = \left(-\frac{1}{4}\, F^I_{\mu\nu} F^{J\,\mu\nu} - \frac{1}{2}\, \partial_\mu \sigma^I\, \partial^\mu \sigma^J + Y^I_{ij} Y^{J\,ij} \right) a_{IJ}(\sigma) - \frac{1}{24}\, \epsilon^{\mu\nu\lambda\rho\sigma}\, A^I_\mu F^J_{\nu\lambda} F^K_{\rho\sigma}\, F_{IJK}\,. \tag{5.1}$$

We use the following conventions: $\mu, \nu = 0,1,2,3,5$ are 5-dimensional vector indices and $m, n, \ldots$ are the corresponding quantities in four dimensions (see appendix A for a summary of conventions). The index $*$ takes the values 0 or 5 for compactifying the time-like and space-like coordinate, respectively. The 5-dimensional gauge potential A^I_μ decomposes into its 4-dimensional counterpart A^I_m and a new scalar field b^I:[23]

$$A^I_\mu \quad \Longrightarrow \quad \begin{cases} \left(A^I_m,\, b^I := A^{I5} = A^I_5 \;\right) & (1,3)\,, \\ \left(A^I_m,\, b^I := A^{I0} = -A^I_0 \right) & (0,4)\,. \end{cases} \tag{5.2}$$

Employing these conventions and (A.8), the terms in (5.1) dimensionally reduce as follows:

$$\begin{aligned} -\tfrac{1}{4}\, F^I_{\mu\nu} F^{J\,\mu\nu} &\Longrightarrow \begin{cases} -\dfrac{1}{4} F^I_{mn} F^{J\,mn} - \dfrac{1}{2}\, \partial_m b^I \partial^m b^J & (1,3)\,, \\ -\dfrac{1}{4} F^I_{mn} F^{J\,mn} + \dfrac{1}{2}\, \partial_m b^I \partial^m b^J & (0,4)\,, \end{cases} \\ -\frac{1}{2}\, \partial_\mu \sigma^I\, \partial^\mu \sigma^J &\Longrightarrow \begin{cases} -\dfrac{1}{2}\, \partial_m \sigma^I\, \partial^m \sigma^J & (1,3)\,, \\ -\dfrac{1}{2}\, \partial_m \sigma^I\, \partial^m \sigma^J & (0,4)\,, \end{cases} \\ -\frac{1}{24}\, \epsilon^{\mu\nu\lambda\rho\sigma} A^{(I}_\mu F^J_{\nu\lambda} F^{K)}_{\rho\sigma} &\Longrightarrow \begin{cases} +\dfrac{1}{8}\, \epsilon^{mnpq} b^{(I} F^J_{mn} F^{K)}_{pq} = +\dfrac{1}{4}\, b^{(I}\, \tilde{F}^J_{mn} F^{K)\,mn} & (1,3)\,, \\ +\dfrac{1}{8}\, \epsilon^{mnpq} b^{(I} F^J_{mn} F^{K)}_{pq} = +\dfrac{1}{4}\, b^{(I}\, \tilde{F}^J_{mn} F^{K)\,mn} & (0,4)\,. \end{cases} \end{aligned} \tag{5.3}$$

The term containing the auxiliary field $Y^{I\,ij}$ reduces trivially, since it does not contain any space-time derivatives. To obtain the result for the Chern-Simons term we integrated by parts and introduced the dual 4-dimensional field strength tensor:

$$\tilde{F}_{mn} := \frac{1}{2}\, \epsilon_{mnpq}\, F^{pq}\,. \tag{5.4}$$

Hence the bosonic sector of the dimensionally reduced lagrangian in 4-dimensional Minkowski-space is then given by

$$\mathcal{L}^{(1,3)}_{\text{bos}} = \left(-\frac{1}{4} F^I_{mn} F^{J\,mn} - \frac{1}{2}\, \partial_m \sigma^I \partial^m \sigma^J - \frac{1}{2}\, \partial_m b^I \partial^m b^J \right) a_{IJ}(\sigma) +$$

[23] We are of course free to choose the sign and normalization of the scalar field b^I. The above choice will turn out to be convenient.

$$+\frac{1}{4} b^I \tilde{F}^J_{mn} F^{K\,mn} F_{IJK} + Y^I_{ij} Y^{J\,ij} a_{IJ}(\sigma)\,. \tag{5.5}$$

For euclidean signature we obtain:

$$\begin{aligned}\mathcal{L}^{(0,4)}_{\text{bos}} &= \left(-\frac{1}{4} F^I_{mn} F^{J\,mn} - \frac{1}{2}\,\partial_m \sigma^I \partial^m \sigma^J + \frac{1}{2}\,\partial_m b^I \partial^m b^J\right) a_{IJ}(\sigma) + \\ &\quad + \frac{1}{4} b^I \tilde{F}^J_{mn} F^{K\,mn} F_{IJK} + Y^I_{ij} Y^{J\,ij} a_{IJ}(\sigma)\,. \end{aligned} \tag{5.6}$$

As was already anticipated in the introduction, the new scalar kinetic term obtained from dimensional reduction of a field strength has a minus sign relative to the kinetic term of σ^I. Hence the metric of the scalar manifold has split signature.

5.1.2 The scalar manifold in the minkowskian case

Let us now discuss the geometry underlying the lagrangian (5.5). It is well known that the corresponding scalar manifold $\mathcal{M}_M$ must be an *affine special Kähler manifold* [34, 35, 36, 37, 67, 38, 39]. We start to make this manifest by introducing complex fields

$$X^I := \sigma^I + i b^I\,, \qquad \bar{X}^I := (X^I)^* = \sigma^I - i b^I\,, \tag{5.7}$$

in terms of which the scalar kinetic term becomes

$$-\frac{1}{2}\left(\partial_m \sigma^I \partial^m \sigma^J + \partial_m b^I \partial^m b^J\right) a_{IJ}(\sigma) = -\frac{1}{2}\,\partial_m \bar{X}^I \partial^m X^J\, a_{IJ}(\sigma)\,. \tag{5.8}$$

Since the matrix $(a_{IJ}(\sigma))$ is real, symmetric and positive definite, we find that the metric appearing in (5.8) is hermitean with respect to the complex structure on the scalar manifold $\mathcal{M}_M$ specified by (5.7). Next we construct a Kähler potential for the metric, which shows that $\mathcal{M}_M$ is Kähler. In this course we first define a new holomorphic prepotential $F(X)$ by replacing the argument of the 5-dimensional real polynomial prepotential $F(\sigma)$ by the new holomorphic coordinate X:

$$F(\sigma) \to F(X)\,, \qquad \text{by substituting } \sigma \to X\,. \tag{5.9}$$

Note that for this substitution to make sense, it is sufficient to assume that $F(\sigma)$ is real analytic, not necessarily polynomial. We further define

$$F_{IJ}(X) := \frac{\partial^2 F(X)}{\partial X^I \partial X^J}\,, \qquad \bar{F}_{IJ}(\bar{X}) := \frac{\partial^2 \bar{F}(\bar{X})}{\partial \bar{X}^I \partial \bar{X}^J}\,, \tag{5.10}$$

where $\bar{F}(\bar{X}) := (F(X))^*$ is the complex conjugate of $F(X)$. Since $F(\sigma)$ is of at most cubic degree, we can explicitly express $F_{IJ}(X)$ in terms of the real fields σ^I, b^I by means of an exact first order Taylor expansion around σ:

$$F_{IJ}(X) = a_{IJ}(\sigma) + i\, F_{IJK}\, b^K\,, \qquad \bar{F}_{IJ}(\bar{X}) = a_{IJ}(\sigma) - i\, F_{IJK}\, b^K\,. \tag{5.11}$$

This relation can then be used to express $a_{IJ}(\sigma)$ in terms of X^I and $\bar{X}^I$:

$$N_{IJ}(X,\bar{X}) := \frac{1}{2}\left(F_{IJ}(X) + \bar{F}_{IJ}(\bar{X})\right) = a_{IJ}(\sigma)\,. \tag{5.12}$$

We see that $N_{IJ}(X,\bar{X})$ has the Kähler potential

$$K(X,\bar{X}) = \frac{1}{2}\left(F_I(X)\bar{X}^I + \bar{F}_I(\bar{X})X^I\right). \tag{5.13}$$

In fact, the Kähler potential is not generic, because it can be expressed in terms of the holomorphic prepotential $F(X)$. Therefore $\mathcal{M}_M$ is not only a Kähler manifold, but an affine special Kähler manifold.

In order to rewrite the remaining terms in (5.5) in terms of the complex fields (5.7), we decompose the field strength into its selfdual and antiselfdual parts:[24]

$$F^I_{\pm|mn} := \frac{1}{2}\left(F^I_{mn} \pm \frac{1}{i}\tilde{F}^I_{mn}\right). \tag{5.14}$$

Employing this decomposition, the gauge kinetic term and the Chern-Simons term combine to:

$$\begin{aligned} -\frac{1}{4}F^I_{mn}F^{J\,mn}\,a_{IJ}(\sigma) + \frac{1}{4}b^I\,\tilde{F}^J_{mn}F^{K\,mn}\,F_{IJK} = &-\frac{1}{4}\,F^I_{-|mn}F^{J|mn}_{-}\,F_{IJ}(X) - \\ &-\frac{1}{4}\,F^I_{+|mn}F^{J|mn}_{+}\,\bar{F}_{IJ}(\bar{X}). \end{aligned} \tag{5.15}$$

This result then completes the rewriting of the lagrangian (5.5) in term of the complex scalar fields (5.7):

$$\begin{aligned} \mathcal{L}^{(1,3)}_{\text{bos}} = &-\frac{1}{4}\,F^I_{-|mn}F^{J|mn}_{-}\,F_{IJ}(X) - \frac{1}{4}\,F^I_{+|mn}F^{J|mn}_{+}\,\bar{F}_{IJ}(\bar{X}) - \\ &-\frac{1}{2}\,\partial_m\bar{X}^I\partial^m X^J\,N_{IJ}(X,\bar{X}) + Y^I_{ij}Y^{J\,ij}\,N_{IJ}(X,\bar{X}). \end{aligned} \tag{5.16}$$

In order to make contact with the more recent literature, and also to the construction of affine special Kähler manifolds given in [39], we now change from the "old conventions" of [55] to the "new conventions" of [72] by rescaling the prepotential:

$$F^{(\text{new})}(X) = \frac{1}{2i}F^{(\text{old})}(X). \tag{5.17}$$

In terms of these conventions the lagrangian (5.16) becomes

$$\begin{aligned} \mathcal{L}^{d=4}_{\text{bos}} = &\frac{i}{2}\,F^I_{+|mn}F^{J|mn}_{+}\,\bar{F}_{IJ}(\bar{X}) - \frac{i}{2}\,F^I_{-|mn}F^{-|mn}_{+}\,F_{IJ}(X) - \\ &-\frac{1}{2}\,\partial_m X^I\partial^m\bar{X}^J\,N_{IJ}(X,\bar{X}) + Y^I_{ij}Y^{J\,ij}\,N_{IJ}(X,\bar{X}), \end{aligned} \tag{5.18}$$

where N_{IJ} is the same as above, i.e.,

$$N_{IJ} = i(F_{IJ} - \bar{F}_{IJ}) \tag{5.19}$$

[24] In Minkowski signature the selfdual and antiselfdual parts are complex, and they are related by complex conjugation. We choose $F^I_{+|mn}$ such that it is selfdual for euclidean signature and require that minkowskian and euclidean expressions take the same form. When comparing to [48, 71] one needs to take into account that their ϵ-tensor is defined by $\epsilon^{0123} = i$, whereas ours is defined by $\epsilon_{0123} = 1$, see also appendix A.

in terms of the new prepotential F, while the Kähler potential (5.13) is replaced by:

$$K(X,\bar{X}) = i\big(F_I(X)\bar{X}^I - \bar{F}_I(\bar{X})X^I\big) = i\begin{pmatrix}\bar{X}^I & \bar{F}_I(\bar{X})\end{pmatrix}\begin{pmatrix}0 & \mathbb{1}\\ -\mathbb{1} & 0\end{pmatrix}\begin{pmatrix}X^I\\ F_I(X)\end{pmatrix}. \qquad (5.20)$$

In order to further explain the geometrical structure of $\mathcal{N}=2$ vector multiplets, let us recall that the scalar fields X^I are the components of a map $\varphi : M = \mathbb{R}^{1,3} \to \mathcal{M}_M$ from Minkowski space-time to an affine special Kähler manifold $\mathcal{M}_M$ with respect to a system of special local coordinates. An intrinsic definition of affine special Kähler manifolds was given in [38]: An *affine special Kähler structure* on a Kähler manifold (M,J,g) is a flat and torsion-free connection ∇, which satisfies (i) and (ii) of Definition 9 in section 2. This definition is equivalent to the definition given in [36, 37, 67], as was shown in [39]. In fact, there is a close analogy with the case of affine special para-Kähler manifolds, which was developed in section 2. Any simply connected affine special Kähler manifold $\mathcal{M}_M$ of complex dimension N admits a (holomorphic) kählerian lagrangian immersion

$$\phi : \mathcal{M}_M \to T^*\mathbb{C}^N\,, \quad p \mapsto \phi(p) = \begin{pmatrix} z^I(\phi(p))\\ w_I(\phi(p))\end{pmatrix}. \qquad (5.21)$$

This immersion is uniquely determined by the special Kähler data (J,g,∇) on $\mathcal{M}_M$ up to a complex affine transformation of $T^*\mathbb{C}^N$ with linear part in $\mathrm{Sp}(2N,\mathbb{R})$. Here (z^I,w_I) are canonical coordinates on $T^*\mathbb{C}^N = \mathbb{C}^{2N}$. Up to an affine transformation as above, we can assume that the functions $\tilde{z}^I := z^I\circ\phi$ provide local holomorphic coordinates (called *special coordinates*) in a neighborhood of a point in $\mathcal{M}_M$. The functions $\tilde{w}_I := w_I\circ\phi$ are then expressed in terms of

$$\tilde{w}_I = F_I(\tilde{z}^1,\dots,\tilde{z}^N)\,, \qquad (5.22)$$

where $F = F(z^1,\dots,z^N)$ is the holomorphic prepotential, which locally generates the holomorphic lagrangian immersion. This shows that

$$\begin{pmatrix} X^I(x)\\ F_I(X(x))\end{pmatrix} = \phi(\varphi(x)) = \begin{pmatrix}\tilde{z}^I(\varphi(x))\\ \tilde{w}_I(\varphi(x))\end{pmatrix}, \qquad (5.23)$$

where $x\in M$ and $\varphi : M \to \mathcal{M}_M$.

If the target manifold $\mathcal{M}_M$ is not simply connected, we can cover it by simply connected open sets U_α, such that we have kählerian lagrangian immersions

$$\phi_\alpha : U_\alpha \to T^*\mathbb{C}^N\,, \qquad p\mapsto \phi_\alpha(p) = \begin{pmatrix} z^I(\phi_\alpha(p))\\ w_I(\phi_\alpha(p))\end{pmatrix} =: \begin{pmatrix} z^I_\alpha(p)\\ w^\alpha_I(p)\end{pmatrix}. \qquad (5.24)$$

These are related by

$$\begin{pmatrix} z^I_\alpha\\ w^\alpha_I\end{pmatrix} = M_{\alpha\beta}\begin{pmatrix} z^I_\beta\\ w^\beta_I\end{pmatrix} + v_{\alpha\beta}\,, \qquad (5.25)$$

where $M_{\alpha\beta}\in\mathrm{Sp}(2N,\mathbb{R})$ and $v_{\alpha\beta}\in\mathbb{C}^{2N}$. One can prove that it is possible to choose the U_α such that the functions $x^I_\alpha := \mathrm{Re}\, z^I_\alpha$ and $y^\alpha_I := \mathrm{Re}\, w^\alpha_I$ define a (real) local affine coordinate system on U_α with respect to the flat torsion-free special connection ∇ [39]. Then the data $(M_{\alpha\beta}, v_{\alpha\beta})$ automatically satisfy a cocycle condition. The real affine symplectic transformations $(M_{\alpha\beta}, \mathrm{Re}\, v_{\alpha\beta})$ are the transitions between the ∇-affine coordinate systems on U_α and U_β. Note that the linear parts $(M_{\alpha\beta})$ are the constant transition functions of the tangent bundle endowed with the flat connection ∇.

The flat torsion-free connection ∇ is part of the intrinsic definition of an affine special Kähler manifold $\mathcal{M}_M$. Any such manifold admits an S^1-family of such special connections $\nabla^t := e^{tJ} \circ \nabla \circ e^{-tJ}$, where J is the complex structure of $\mathcal{M}_M$ and $\circ$ denotes composition, i.e.,

$$\nabla^t_X Y := e^{tJ} \nabla_X (e^{-tJ} Y) \qquad \text{for all vector fields } X, Y \,. \tag{5.26}$$

If we do not fix the connection ∇ then the immersions ϕ_α are only unique up to a complex affine transformation with linear part in $\mathrm{U}(1) \cdot \mathrm{Sp}(2N, \mathbb{R})$. This explains the additional phase factor in the transition functions of [36, 37, 67].

If we choose U_α sufficiently small, such that the immersion ϕ_α is an embedding, then we can identify U_α with its image: $U_\alpha \simeq \phi_\alpha(U_\alpha) \subset T^*\mathbb{C}^N$. The embedding ϕ_α provides us with $2N$ holomorphic functions (z^I_α, w^α_J) on $U_\alpha \subset \mathcal{M}_M$, such that a point $p \in U_\alpha$ is completely determined by the values of these functions at p. Moreover, by further restricting U_α if necessary, we can choose N of these functions to define a global holomorphic coordinate system on U_α. If the submanifold $\phi_\alpha(U_\alpha) \subset T^*\mathbb{C}^N$ is transverse to the fibers of the bundle $\pi : T^*\mathbb{C}^N \to \mathbb{C}^N$, then the z^I_α can be taken as holomorphic coordinates of $\mathcal{M}_M$, as already mentioned. Geometrically, this corresponds to projecting the submanifold onto the directions of the coordinates z^I of $T^*\mathbb{C}^N$ via the projection π. From this picture it is immediately clear that there are also non-generic situations where the submanifold has a vertical tangent vector at some point p, so that the z^I are not independent, and cannot be used as coordinates near p. In field theory this corresponds to situations where the scalar fields X^I are not independent, and where the lower components F_J of the vector $(X^I, F_J)^T$ are not the components of the gradient of a function F [68]. This is then called a 'symplectic basis without a prepotential.' Therefore it is often advantageous to work in terms of the embedding coordinates (also called symplectic vectors) $(\tilde{z}^I, \tilde{w}_J)$ (X^I, F_J), *etc.*, which are provided naturally by the extrinsic construction. Alternatively, one can always perform a linear symplectic transformation of $T^*\mathbb{C}^N$ which makes the immersion transverse to the fibers of $\pi : T^*\mathbb{C}^N \to \mathbb{C}^N$ and the upper components $\tilde{z}^I$ independent, in a neighborhood of a given point $p \in U_\alpha$. In other words, there is always a symplectic basis where a prepotential exists [37, 69].

We now turn to the gauge fields. As is well known, the $\mathrm{Sp}(2N, \mathbb{R})$ transformations act on them as generalized electric-magnetic duality transformations [70]. In fact, these duality rotations are responsible for the additional geometric structures on the scalar manifold $\mathcal{M}_M$, which make it special Kähler rather than just Kähler. To specify the action of $\mathrm{Sp}(2N, \mathbb{R})$ on the gauge fields one defines dual gauge fields by $G_{-I|mn} := F_{IJ} F^J_{-|mn}$ and $G_{+I|mn} := G^*_{-I|mn}$. Then the combined Bianchi identities and Euler-Lagrange equations take the form

$$\partial^m \begin{pmatrix} F^I_{+|mn} - F^I_{-|mn} \\ G_{+J|mn} - G_{-J|mn} \end{pmatrix} = 0 \,. \tag{5.27}$$

As such, the equations are invariant under $\mathrm{GL}(2N,\mathbb{R})$. But the fields $G_{\pm I|mn}$ are not independent of the $F^I_{\pm|mn}$. In fact, they are completely determined by the gauge fields and the scalars, and therefore transformations of the $G_{\pm I|mn}$ must be induced by transformations of the independent fields. Moreover, we are working within the lagrangian formulation of field theory. Hence the transformed equations are required to be the Bianchi identities and Euler-Lagrange equations of a lagrangian of the form (5.18). For a generic prepotential F this implies that one can only make $\mathrm{Sp}(2N,\mathbb{R})$ transformations (or rescalings, which are not interesting because they do not mix different gauge fields). Moreover F_{IJ} must also transform under $\mathrm{Sp}(2N,\mathbb{R})$, with a transformation law, which is precisely induced by a linear symplectic transformation of the vector $(X^I, F_J(X))^T$. The action of $\mathrm{Sp}(2N,\mathbb{R})$ extends to the full $\mathcal{N}=2$ supersymmetric equations of motion, including the fermions, if one defines the dual gauge fields as $G_{-I|mn} = \delta S/\delta F^{-I|mn} = F_{IJ}F^J_{-|mn} + O_{-I|mn}$, where $O_{-I|mn}$ are those fermionic terms in the action which couple to $F^I_{-|mn}$.

Notice that $(F^I_{-|mn}, G_{-J|mn})^T$ is of the form $(V^I, F_{JK}V^K)^T$ and, hence, is tangent, along the map $\phi_\alpha \circ \varphi$, to the lagrangian submanifold $\phi_\alpha(U_\alpha) \subset T^*\mathbb{C}^N$, defined by the prepotential F. In other words, for fixed $x \in M$, the vector $(F^I_{-|mn}(x), G_{-J|mn}(x))^T$ is tangent to the submanifold $\phi_\alpha(U_\alpha) \subset T^*\mathbb{C}^N$ at the point $\phi_\alpha(\varphi(x))$. It corresponds to a tangent vector of $\mathcal{M}_M$ at $\varphi(x) \in U_\alpha \subset \mathcal{M}_M$. The corresponding vector field is a local section of the bundle $\varphi^*(T\mathcal{M}_M) \to M$, the pullback by $\varphi : M \to \mathcal{M}_M$ of the tangent bundle $T\mathcal{M}_M \to \mathcal{M}_M$.

Recall that we are assuming that the holomorphic functions z^I, which correspond to the scalar fields X^I, are independent on the submanifold $\phi_\alpha(U_\alpha) \subset T^*\mathbb{C}^N$ and therefore provide holomorphic coordinates $z^I_\alpha = z^I \circ \phi_\alpha$ on $U_\alpha \subset \mathcal{M}_M$. The components of the tangent vector $(F^I_{-|mn}(x), G_{-J|mn}(x))^T$ with respect to the coordinate vector fields $\partial/\partial z^I_\alpha$ are precisely the $F^I_{-|mn}(x)$, $I = 1, \ldots, N$.

The vector $(F^I_{+|mn}(x), G_{+J|mn}(x))^T \in T^*\mathbb{C}^N$ is perpendicular to the submanifold $\phi_\alpha(U_\alpha) \subset T^*\mathbb{C}^N$ at the point $\phi_\alpha(\varphi(x))$ with respect to the canonical pseudo-hermitean metric $\gamma := i\Omega(\cdot, \bar{\cdot})$, defined by the complex symplectic form Ω and the complex conjugation on $T^*\mathbb{C}^N$. This follows from the fact that $(F^I_{+|mn}(x), G_{+J|mn}(x))^T$ is the complex conjugate of the tangent vector $(F^I_{-|mn}(x), G_{-J|mn}(x))^T$.

5.1.3 The euclidean scalar manifold in terms of adapted coordinates

In the euclidean signature case (5.6) we can perform an analogous construction. Here we define real fields X^I_+ and X^I_-:

$$X^I_+ := \sigma^I + b^I \,, \qquad X^I_- := \sigma^I - b^I \,. \tag{5.28}$$

As these correspond to the adapted coordinates $z^i_\pm$ introduced in section 2, we will refer to them as "adapted coordinates." The scalar kinetic term now takes the form

$$-\frac{1}{2}\left(\partial_m \sigma^I \partial^m \sigma^J - \partial_m b^I \partial^m b^J\right) a_{IJ}(\sigma) = -\frac{1}{2}\,\partial_m X^I_- \partial^m X^J_+ \, a_{IJ}(\sigma) \,. \tag{5.29}$$

In order to rewrite all other quantities appearing in the lagrangian (5.6) in terms of adapted coordinates, we now carry out a construction similar to the one used for minkowskian signature. In this course we first introduce new prepotentials $F^+(X_+)$ and $F^-(X_-)$ by replacing the argument of the real-valued polynomial prepotential $F(\sigma)$ by X_+ and X_-, respectively:

$$\begin{aligned} F(\sigma) &\to F^+(X_+)\,, &\quad& \text{substituting: } \sigma \to X_+\,, \\ F(\sigma) &\to F^-(X_-)\,, && \text{substituting: } \sigma \to X_-\,. \end{aligned} \tag{5.30}$$

Note that the substitution makes sense for any real-valued function $F(\sigma)$ and that $F^+(X_+)$ and $F^-(X_-)$ are again *real-valued* functions. Analogous to eq. (5.10), we also define $F^+_{IJ}(X_+)$ and $F^-_{IJ}(X_-)$ as

$$F^+_{IJ}(X_+) := \frac{\partial^2 F^+(X_+)}{\partial X^I_+ \partial X^J_+}\,, \qquad F^-_{IJ}(X_-) := \frac{\partial^2 F^-(X_-)}{\partial X^I_- \partial X^J_-}\,. \tag{5.31}$$

Since the prepotential is a polynomial of degree at most 3, we can relate $F^+_{IJ}(X_+)$ and $F^-_{IJ}(X_-)$ to the real scalar fields σ^I and b^I by:

$$F^+_{IJ}(X_+) = a_{IJ}(\sigma) + F_{IJK}\, b^K\,, \qquad F^-_{IJ}(X_-) = a_{IJ}(\sigma) - F_{IJK}\, b^K\,. \tag{5.32}$$

The metric $a_{IJ}(\sigma)$ may then be rewritten as

$$N_{IJ}(X_+, X_-) := \frac{1}{2}\left(F^+_{IJ}(X_+) + F^-_{IJ}(X_-)\right) = a_{IJ}(\sigma)\,. \tag{5.33}$$

Analogously to the minkowskian case, we also obtain a potential

$$K(X_+, X_-) = \frac{1}{2}\left(F^+_I(X_+) X^I_- + F^-_I(X_-) X^I_+\right), \tag{5.34}$$

which satisfies:

$$N_{IJ}(X_+, X_-) = \frac{\partial^2 K(X_+, X_-)}{\partial X^I_+ \partial X^J_-}\,. \tag{5.35}$$

Thus $K(X_+, X_-)$ is a potential for N_{IJ}, which can be obtained from a prepotential $F^+(X_+)$, which only depends on X_+ but not on X_-. In fact, the other function $F^-(X_-)$ which enters the definition of $K(X_+, X_-)$ is obtained by simply replacing the argument of $F^+(X_+)$ by X_-. As will become clear later, this reflects the fact that the underlying para-holomorphic prepotential is not the most general one, but is real-valued on real points, since it comes from the real-valued prepotential $F(\sigma)$. A general para-holomorphic prepotential can be described in terms of two independent real-valued prepotentials $F^+(X_+)$ and $F^-(X_-)$, see 2.

Comparing to section 2 we see that the scalar manifold $\mathcal{M}_E$ is an *(affine) special para-Kähler manifold*, parametrized in terms of adapted real coordinates. Hence we can also parameterize the manifold by para-holomorphic coordinates. This will be done in the next section, where we will also discuss the relation to section 2 in more detail.

We conclude this section by rewriting the terms containing the field strength in terms of the adapted coordinates X^I_+, X^I_-. To this end we introduce real selfdual and antiselfdual field strength tensors according to:

$$F^{\mathrm{E}\,I}_{\pm|mn} := \frac{1}{2}\big(F^I_{mn} \pm \tilde{F}^I_{mn}\big)\,. \tag{5.36}$$

Substituting this decomposition into the gauge kinetic term and Chern-Simons term of eq. (5.6), we obtain:

$$-\frac{1}{4}F^I_{mn}F^{J\,mn}\,a_{IJ}(\sigma)+\frac{1}{4}\,b^I\,\tilde{F}^J_{mn}F^{K\,mn}\,F_{IJK}\;=\;=-\frac{1}{4}\,F^{\mathrm{E}\,I}_{-|mn}F^{\mathrm{E}\,J|mn}_{-}\,F^+_{IJ}(X_+)-$$
$$-\frac{1}{4}\,F^{\mathrm{E}\,I}_{+|mn}F^{\mathrm{E}\,J|mn}_{+}\,F^-_{IJ}(X_-)\,. \qquad (5.37)$$

Combining the terms derived in this subsection we are then in a position to write down the bosonic lagrangian (5.6) in terms of adapted coordinates:

$$\mathcal{L}^{(0,4)}_{\text{bos}}\;=\;-\frac{1}{4}\,F^{\mathrm{E}\,I}_{-|mn}F^{\mathrm{E}\,J|mn}_{-}\,F^+_{IJ}(X_+)-\frac{1}{4}\,F^{\mathrm{E}\,I}_{+|mn}F^{\mathrm{E}\,J|mn}_{+}\,F^-_{IJ}(X_-)-$$
$$-\frac{1}{2}\,\partial_m X^I_-\partial^m X^J_+\,N_{IJ}(X_+,X_-)+Y^I_{ij}Y^{J\,ij}\,N_{IJ}(X_+,X_-)\,. \qquad (5.38)$$

5.1.4 The euclidean scalar manifold in terms of para-complex coordinates

In section 2 we found that para-complex manifolds can be parametrized by using either adapted coordinates $z^i_\pm$ (as we did in the last subsection) or para-complex coordinates $z^i,\bar{z}^i$. Having studied the adapted coordinates in the last section, we now introduce para-complex fields

$$X^I:=\sigma^I+eb^I\,,\qquad \bar{X}^I:=\sigma^I-eb^I\,. \qquad (5.39)$$

Here e is the para-complex unit number introduced in 2, which satisfies $e^2=1$, $e\neq\mathbb{1}$ and $\overline{e}=-e$.

For these fields we now carry out a construction similar to the one in eqs. (5.9) to (5.13). We first define a para-holomorphic prepotential $F(X)$ by:

$$F(\sigma)\to F(X)\,,\qquad \text{by substituting } \sigma\to X\,. \qquad (5.40)$$

This substitution makes sense for any real-analytic function $F(\sigma)$. The first and second derivatives of $F(X)$ with respect to one of its arguments are again denoted by $F_I(X)$ and $F_{IJ}(X)$. In the case where $F(\sigma)$ is a polynomial of degree at most 3, we can again set:

$$F_{IJ}(X)=a_{IJ}(\sigma)+e\,F_{IJK}\,b^K\,,\qquad \bar{F}_{IJ}(\bar{X})=a_{IJ}(\sigma)-e\,F_{IJK}\,b^K\,. \qquad (5.41)$$

The relation to the metric on the scalar manifold $\mathcal{M}_E$ is given by:

$$N_{IJ}(X,\bar{X}):=\frac{1}{2}\left(F_{IJ}(X)+\bar{F}_{IJ}(\bar{X})\right)=a_{IJ}(\sigma)\,. \qquad (5.42)$$

Comparing eq. (5.42) to (5.12), we can then immediately write down a para-Kähler potential for the metric,

$$K(X,\bar{X})=\frac{1}{2}\left(F_I(X)\bar{X}^I+\bar{F}_I(\bar{X})X^I\right). \qquad (5.43)$$

This shows again that $\mathcal{M}_E$ is a para-Kähler manifold and since the para-Kähler potential comes from a para-holomorphic prepotential $F(X)$, it is in fact an *(affine) special para-Kähler manifold.* Note that the potential (5.34) is also a para-Kähler potential, since $\partial^2/\partial X^I_+\partial X^J_-=\partial^2/\partial X^I\partial\bar{X}^J$. In fact, using the decomposition

$$F(X)=\frac{1}{2}(F^+(X_+)+F^-(X_-))+e\frac{1}{2}(F^+(X_+)-F^-(X_-))\,, \qquad (5.44)$$

one can easily check that both potentials coincide: $K(X_+,X_-)=K(X,\bar{X})$.

In order to completely rewrite the lagrangian (5.6) in terms of para-complex fields, we also introduce para-complex (anti-)selfdual field strength tensors

$$F^I_{\pm|mn} := \frac{1}{2}\left(F^I_{mn} \pm \frac{1}{e}\tilde{F}^I_{mn}\right) \tag{5.45}$$

and $\tilde{F}^I_{\pm|mn}$ defined by the analogous formula. These satisfy [25]:

$$\tilde{F}^I_{\pm|mn} = \pm e\, F^I_{\pm|mn}\,, \qquad (F^I_{\pm|mn})^* = F^I_{\mp|mn}\,. \tag{5.46}$$

The para-complex version of the euclidean lagrangian (5.6) is:

$$\begin{aligned}\mathcal{L}^{(0,4)}_{\text{bos}} &= -\frac{1}{4}\, F^I_{-|mn} F^{J|mn}_{-}\, F_{IJ}(X) - \frac{1}{4}\, F^I_{+|mn} F^{J|mn}_{+}\, \bar{F}_{IJ}(\bar{X}) - \\ &\quad -\frac{1}{2}\, \partial_m \bar{X}^I \partial^m X^J\, N_{IJ}(X,\bar{X}) + Y^I_{ij} Y^{J\,ij}\, N_{IJ}(X,\bar{X})\,. \end{aligned} \tag{5.47}$$

Comparing this result to the minkowskian lagrangian given in eq. (5.16) we see that both lagrangians take the same form when written in (para-)holomorphic coordinates.

Again it is useful to redefine the prepotential:

$$F^{(\text{new})}(X) = \frac{1}{2\,e} F^{(\text{alt})}(X)\,. \tag{5.48}$$

With this rescaling our lagrangian (5.47) becomes

$$\begin{aligned}\mathcal{L}^{(0,4)}_{\text{bos}} &= \frac{e}{2}\, F^I_{+|mn} F^{J|mn}_{+}\, \bar{F}_{IJ}(\bar{X}) - \frac{e}{2}\, F^I_{-|mn} F^{J|mn}_{-}\, F_{IJ}(X) - \\ &\quad -\frac{1}{2}\, \partial_m X^I \partial^m \bar{X}^J\, N_{IJ}(X,\bar{X}) + Y^I_{ij} Y^{J\,ij}\, N_{IJ}(X,\bar{X})\,, \end{aligned} \tag{5.49}$$

where

$$N_{IJ}(X,\overline{X}) = \frac{\partial^2 K(X,\bar{X})}{\partial X^I \partial \overline{X}^J} \tag{5.50}$$

is a para-Kähler metric with para-Kähler potential

$$K(X,\bar{X}) = e\left(\bar{X}^I \;\; \bar{F}_I(\bar{X})\right)\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}\begin{pmatrix} X^I \\ F_I(X) \end{pmatrix}. \tag{5.51}$$

We now connect this result to the mathematical description given in section 2. In this course we proceed in the same way as when relating the minkowskian theory to [39]: the para-holomorphic scalar fields X^I are the components of a map $\varphi : E = \mathbb{R}^{0,4} \to \mathcal{M}_E$ from euclidean 4-space into an (affine) special para-Kähler manifold $\mathcal{M}_M = \bigcup_\alpha U_\alpha$. This manifold can be locally immersed into the cotangent bundle T^*C^N of the para-complex vector space C^N by $\phi_\alpha : U_\alpha \to T^*C^N$. If the immersion is generic, it induces local para-holomorphic coordinates $z^I_\alpha = z^I \circ \phi_\alpha$ (and also local adapted coordinates $z^I_{\pm|\alpha}$) and we have $X^I(x) = z^I_\alpha \circ \varphi(x)$ (and $X^I_\pm(x) = z^I_{\pm|\alpha} \circ \varphi(x)$).

[25] So far, the symbol $*$ denoted the usual complex conjugation, while para-complex conjugation was denoted by a $\bar{\cdot}$. Here and in the following we use $*$ to denote the para-complex conjugation, in order to emphasize the analogy of the two geometries.

Through (5.50) and (5.51) the prepotential $F(X)$ determines the para-hermitean form

$$N_{IJ}(z_\alpha, \bar{z}_\alpha) dz^I_\alpha \otimes d\bar{z}^J_\alpha = \left(2\,\mathrm{Im} \frac{\partial^2 F}{\partial z^I_\alpha \partial z^J_\alpha} \right) dz^I_\alpha \otimes d\bar{z}^J_\alpha \,, \tag{5.52}$$

which equals (up to an overall sign)[26] the para-hermitean form $\gamma = \phi^*_\alpha \gamma_V$ induced by the immersion $\phi_\alpha = \phi_F : U_\alpha \to V = T^* C^N$, see (2.3).

Thus we see that all the structures of the bosonic part of the minkowskian theory carry over to the euclidean theory, by just replacing holomorphic by para-holomorphic quantities. In particular symplectic transformations play the same role in both theories. A difference occurs when one does not fix the special connection ∇, but considers families of such connections, which are generated by conjugation with e^{tJ} and e^{tI}. Here J and I are the complex and para-complex structure of $\mathcal{M}_M$ and $\mathcal{M}_E$, respectively. In both cases the structure generates an abelian group, which is compact for J, but non-compact for I. This reflects itself in the symmetries of the (para-)Kähler potential: while (5.20) is invariant under phase transformations $(X^I, F_I)^T \to \exp(i\alpha)(X^I, F_I)^T$, eq. (5.51) is invariant under para-complex phase transformations $(X^I, F_I)^T \to \pm \exp(e\alpha)(X^I, F_I)^T$. The corresponding groups are U(1) and SO(1,1). In the language of [67] this implies a change in the abelian factor of the structure group of the affine bundle characterizing the special geometry. We have already seen that a similar replacement happens for the R-symmetry groups of the underlying supersymmetry algebras. As we will see in more detail in section 5.5, this is intimately related to the fact that the euclidean scalar geometry has to be para-complex rather than complex.

5.2 The supersymmetry variations

After reducing the bosonic sector of (4.13), let us proceed to the 5-dimensional supersymmetry variations (4.6). In their reduction we utilize the dimensional reduction of the $(1,4)$ dimensional Clifford algebra, which was presented in section 3.3.

From experience with $\mathcal{N} = 2$ supersymmetry, we expect that the supersymmetry variations decompose into a "holomorphic" and an "antiholomorphic" piece. Like the gauge fields, the spinors λ^I carry an index of the scalar target manifold and thus have a geometrical interpretation as tangent vectors. One therefore expects that the introduction of (para-)holomorphic or adapted coordinates will induce a "chiral decomposition" of the fermions, as was already anticipated in section 3.3.

In order to identify the projectors of this decomposition, we consider the bosonic part of the supersymmetry variations (4.6). Employing eq. (5.2), the dimensional reduction of the scalar and vector fields yields:

$$\delta A^I_m = \frac{1}{2}\,\bar{\epsilon}\gamma_m \lambda^I \,, \qquad \delta\sigma^I = \frac{i}{2}\,\bar{\epsilon}\lambda^I \,, \qquad \delta b^I = \begin{cases} \frac{1}{2}\,\bar{\epsilon}\gamma^5 \lambda^I & (1,3)\,, \\ \frac{1}{2}\,\bar{\epsilon}\gamma^0 \lambda^I & (0,4)\,. \end{cases} \tag{5.53}$$

[26] In order to obtain the same sign, it suffices to define $\gamma_V := -e\Omega(\cdot, \tau\cdot)$ instead of $\gamma_V := e\Omega(\cdot, \tau\cdot)$, cf. section 2.

Rewriting these variations in terms of the (para-)holomorphic and adapted coordinates, (5.7), (5.39) and (5.28), we obtain:

$$\begin{aligned}
\delta X^I &= \frac{i}{2}\,\bar\epsilon\big(\mathbb{1}+\gamma^5\big)\lambda^I\,, & \delta\bar X^I &= \frac{i}{2}\,\bar\epsilon\left(\mathbb{1}-\gamma^5\right)\lambda^I\,, \quad (1,3)\,,\\
\delta X^I &= \frac{i}{2}\,\bar\epsilon\left(\mathbb{1}-i\,e\gamma^0\right)\lambda^I\,, & \delta\bar X^I &= \frac{i}{2}\,\bar\epsilon\left(\mathbb{1}+i\,e\gamma^0\right)\lambda^I\,, \quad (0,4)\,,\\
\delta X^I_+ &= \frac{i}{2}\bar\epsilon\big(\mathbb{1}-i\gamma^0\big)\lambda^I, & \delta X^I_- &= \tfrac{i}{2}\,\bar\epsilon\big(\mathbb{1}+i\gamma^0\big)\lambda^I\,, \quad (0,4)\,.
\end{aligned} \tag{5.54}$$

This motivates introducing the following matrices:

$$\Gamma_*^{\mathrm{M}} := \gamma^5\,, \qquad \Gamma_* := -i\,e\,\gamma^0\,, \qquad \Gamma_*^{\mathrm{E}} := -i\gamma^0\,. \tag{5.55}$$

The relation between these γ-matrices and the Clifford algebra in the 5-dimensional theory was worked out at the beginning of section 3.3. Note that all three, Γ_*^{M}, Γ_* and Γ_*^{E}, square to $+\mathbb{1}$. Additionally, Γ_*^{M} and Γ_*^{E} are hermitean, while Γ_* is hermitean with respect to the complex structure and anti-hermitean with respect to the para-complex structure. Since they anticommute with the γ-matrices forming the respective 4-dimensional Clifford algebra, we can use them to define chiral projectors:

$$\begin{aligned}
\Gamma_\pm &:= \frac{1}{2}\left(\mathbb{1}\pm\Gamma_*^{\mathrm{M}}\right) \quad (1,3)\,,\\
\Gamma_\pm &:= \frac{1}{2}\left(\mathbb{1}\pm\Gamma_*\right) \quad (0,4)\,,\text{w.r.t. para-complex coordinates}\,,\\
\Gamma_\pm^{\mathrm{E}} &:= \frac{1}{2}\left(\mathbb{1}\pm\Gamma_*^{\mathrm{E}}\right) \quad (0,4)\,,\text{ w.r.t. adapted coordinates}\,.
\end{aligned} \tag{5.56}$$

The relation to para-complex and adapted coordinates will become explicit in equation (5.60) below. Using these projectors, we decompose our spinors according to

$$\begin{aligned}
\lambda &= \lambda_+ + \lambda_-\,, \qquad \lambda_\pm := \Gamma_\pm\lambda \quad (1,3),(0,4)\text{ w.r.t. (para-)holomorphic coordinates}\\
\lambda &= \xi_+ + \xi_-\,, \qquad \xi_\pm := \Gamma_\pm^{\mathrm{E}}\lambda \quad (0,4)\text{ w.r.t. adapted coordinates}\,.
\end{aligned} \tag{5.57}$$

Here and henceforth we will use $\lambda_\pm$, $\epsilon_\pm$ to denote the chiral components of λ and ϵ with respect to (para-)complex coordinates, while we use $\xi_\pm$ and $\eta_\pm$ for the chiral projections of λ and ϵ with respect to adapted coordinates.

Let us briefly comment on this decomposition. First we observe that in the euclidean case $\Gamma_\pm$ becomes $\Gamma_\mp$ under the combined hermitean conjugation with respect to both the complex and the para-complex structure, while in the minkowskian case the projectors are invariant:

$$\left(\Gamma_\pm\right)^\dagger = \begin{cases}\Gamma_\pm & (1,3)\,,\\ \Gamma_\mp & (0,4)\,.\end{cases} \tag{5.58}$$

Looking at the projected symplectic Majorana conditions (3.51), (3.56), we see that by introducing an explicit factor e into the projector we have managed to write the reality constraint for spinors in a uniform way:

$$(\lambda^i_\pm)^* = B\epsilon_{ij}\lambda^j_\mp\,, \qquad B = C\gamma_0 = -iC\Gamma_*^E\,, \tag{5.59}$$

where in the euclidean case '$*$' denotes simultaneous complex and para-complex conjugation. Thus by introducing para-complex valued chiral projections we can compensate for the fact that standard chiral projections in euclidean signatures are real, in the sense that $(\xi^i_\pm)^* = B\epsilon_{ij}\xi^j_\pm$.

Of cause we also have to check that $\Gamma_\pm$ project onto complementary subspaces for euclidean signature. In terms of the spinors (5.57) the euclidean projector $\Gamma_\pm$ has eigenvalues $(1+e)$ and $(1-e)$. This can also be deduced from the identification $\Gamma_* \equiv e\,\Gamma^{\rm E}_*$. This looks peculiar, but, due to $(1+e)(1-e)=0$, the projector identity $\Gamma_\pm\Gamma_\mp = 0$ still holds. At this point it is crucial that the ring of para-complex numbers has zero divisors, in order for $\Gamma_\pm$ being a well-defined projector.

Having established the decomposition (5.57), we now rewrite the supersymmetry variations (5.54) in terms of the chirally projected spinors:

$$\begin{aligned} \delta X^I &= i\,\bar\epsilon\,\Gamma_+\lambda^I = i\,\bar\epsilon_+\,\lambda^I_+\,, \qquad & \delta \bar X^I &= i\,\bar\epsilon\,\Gamma_-\lambda^I = i\,\bar\epsilon_-\,\lambda^I_-\,,\\ \delta X^I_+ &= i\,\bar\epsilon\,\Gamma^{\rm E}_+\lambda^I = i\,\bar\eta_+\,\xi^I_+\,, \qquad & \delta X^I_- &= i\,\bar\epsilon\,\Gamma^{\rm E}_-\lambda^I = i\,\bar\eta_-\,\xi^I_-\,. \end{aligned} \tag{5.60}$$

Here the first line holds for both signatures in terms of complex and para-complex quantities, respectively. We further observe that the supersymmetry variations indeed split into a holomorphic and antiholomorphic sector.

It is now straightforward to reduce the remaining supersymmetry variations and to rewrite them in terms of (para-)holomorphic and adapted coordinates, respectively. We start with the 5-dimensional supersymmetry variations of the spinor fields $\delta\lambda^{iI} = -\frac{1}{4}\,\gamma^{\mu\nu}\times F^I_{\mu\nu}\epsilon^i - \frac{i}{2}\,\partial\!\!\!/\,\sigma^I\epsilon^i - Y^{ij\,I}\epsilon_j$. Using the identity

$$\gamma^{\mu\nu}F^I_{\mu\nu} = \begin{cases} \gamma^{mn}F^I_{mn} + 2\gamma^m\gamma^5\partial_m b^I = \gamma^{mn}F^I_{mn} + 2\gamma^m\Gamma^{\rm M}_*\,\partial_m b^I & (1,3)\,,\\ \gamma^{mn}F^I_{mn} - 2\gamma^m\gamma^0\partial_m b^I = \gamma^{mn}F^I_{mn} - 2ie\gamma^m\Gamma_*\,\partial_m b^I & (0,4)\,,\\ \gamma^{mn}F^I_{mn} - 2\gamma^m\gamma^0\partial_m b^I = \gamma^{mn}F^I_{mn} - 2i\gamma^m\Gamma^{\rm E}_*\,\partial_m b^I & (0,4)\,, \end{cases} \tag{5.61}$$

one observes that the reduced terms containing the scalars σ^I and b^I combine into the fields X^I, $\bar X^I$ and X^I_+, X^I_- as follows:

$$\begin{aligned} -\frac{1}{2}\,\gamma^m\partial_m b^I\,\Gamma^{\rm M}_* - \frac{i}{2}\,\gamma^m\partial_m\sigma^I &= -\frac{i}{2}\left(\partial\!\!\!/\bar X^I\Gamma_+ + \partial\!\!\!/X^I\Gamma_-\right) \quad (1,3)\,,\\ +\frac{ie}{2}\,\gamma^m\partial_m b^I\,\Gamma_* - \frac{i}{2}\,\gamma^m\partial_m\sigma^I &= -\frac{i}{2}\left(\partial\!\!\!/\bar X^I\Gamma_+ + \partial\!\!\!/X^I\Gamma_-\right) \quad (0,4)\,,\\ +\frac{i}{2}\,\gamma^m\partial_m b^I\,\Gamma^{\rm E}_* - \frac{i}{2}\,\gamma^m\partial_m\sigma^I &= -\frac{i}{2}\left(\partial\!\!\!/X^I_-\Gamma^{\rm E}_+ + \partial\!\!\!/X^I_+\Gamma^{\rm E}_-\right) \quad (0,4)\,. \end{aligned} \tag{5.62}$$

The dimensional reduction of the auxiliar field $Y^{I\,ij}$ is trivial. The only change that occurs in its supersymmetry variations is $\partial\!\!\!/ = \gamma^\mu\,\partial_\mu \longrightarrow \partial\!\!\!/ = \gamma^m\,\partial_m$, since the spinors λ^i no longer depend on the coordinate x^*. Using the identity (A.10) one can further check that $\gamma^{mn}F^I_{mn}$ can be decomposed into (anti-)selfdual terms according to

$$\gamma^{mn}F^I_{mn} = \gamma^{mn}F^I_{-|mn}\,\Gamma_+ + \gamma^{mn}F^I_{+|mn}\,\Gamma_-\,, \tag{5.63}$$

which holds for all three (anti-)selfdual field strength tensors (5.14), (5.45) and (5.36) and the corresponding projectors (5.56), respectively.

We can now write down the complete dimensionally reduced supersymmetry variations. For (para-) holomorphic fields they can be uniformly written as:

$$\begin{aligned}
\delta X^I &= i\,\bar\epsilon_+\,\lambda^I_+\,,\\
\delta \bar X^I &= i\,\bar\epsilon_-\,\lambda^I_-\,,\\
\delta\lambda^{iI}_+ &= -\frac{1}{4}\gamma^{mn}F^I_{-\,mn}\epsilon^i_+ - \frac{i}{2}\,\partial\!\!\!/ X^I\epsilon^i_- - Y^{ij\,I}\epsilon_{+\,j}\,,\\
\delta\lambda^{iI}_- &= -\frac{1}{4}\gamma^{mn}F^I_{+\,mn}\epsilon^i_- - \frac{i}{2}\,\partial\!\!\!/ \bar X^I\epsilon^i_+ - Y^{ij\,I}\epsilon_{-\,j}\,,\\
\delta A^I_m &= \frac{1}{2}\left(\bar\epsilon_+\gamma_m\lambda^I_- + \bar\epsilon_-\gamma_m\lambda^I_+\right),\\
\delta Y^{ij\,I} &= -\frac{1}{2}\left(\bar\epsilon^{(i}_+\partial\!\!\!/\lambda^{j)\,I}_- + \bar\epsilon^{(i}_-\partial\!\!\!/\lambda^{j)\,I}_+\right).
\end{aligned}\tag{5.64}$$

For completeness, we also give the supersymmetry variations in terms of the adapted coordinates X^I_+, X^I_- and the corresponding chirally projected spinors $\xi_\pm$. These read:

$$\begin{aligned}
\delta X^I_+ &= i\,\bar\eta_+\,\xi^I_+\,,\\
\delta X^I_- &= i\,\bar\eta_-\,\xi^I_-\,,\\
\delta\xi^{iI}_+ &= -\frac{1}{4}\gamma^{\mathrm{E}\,mn}F^{\mathrm{E}\,I}_{-|mn}\eta^i_+ - \frac{i}{2}\,\partial\!\!\!/ X^I_+\eta^i_- - Y^{ij\,I}\eta_{+\,j}\,,\\
\delta\xi^{iI}_- &= -\frac{1}{4}\gamma^{\mathrm{E}\,mn}F^{\mathrm{E}\,I}_{+|mn}\eta^i_- - \frac{i}{2}\,\partial\!\!\!/ X^I_-\eta^i_+ - Y^{ij\,I}\eta_{-\,j}\,,\\
\delta A^I_m &= \frac{1}{2}\left(\bar\eta_+\gamma_m\xi^I_- + \bar\eta_-\gamma_m\xi^I_+\right),\\
\delta Y^{ij\,I} &= -\frac{1}{2}\left(\bar\eta^{(i}_+\partial\!\!\!/\xi^{j)\,I}_- + \bar\eta^{(i}_-\partial\!\!\!/\xi^{j)\,I}_+\right).
\end{aligned}\tag{5.65}$$

5.3 The fermionic sector

We now turn to the fermionic part of the 5-dimensional lagrangian (4.13)

$$\mathcal{L}^{(1,4)}_{\text{ferm}} = -\frac{1}{2}\,\bar\lambda^I\partial\!\!\!/\,\lambda^J\,a_{IJ}(\sigma) - \frac{i}{8}\,\bar\lambda^I\gamma^{\mu\nu}F^J_{\mu\nu}\,\lambda^K F_{IJK} - \frac{i}{2}\,\bar\lambda^{iI}\lambda^{jJ}\,Y^K_{ij}\,F_{IJK}\,.\tag{5.66}$$

With the results of the previous section is now straightforward to reduce (5.66). The corresponding terms become:

$$-\frac{1}{2}\bar\lambda^I\partial\!\!\!/\lambda^J\,a_{IJ}(\sigma) \longrightarrow -\frac{1}{2}\bar\lambda^I\partial\!\!\!/\lambda^J\,a_{IJ}(\sigma) = -\frac{1}{2}\bar\lambda^I\partial\!\!\!/\lambda^J\,a_{IJ}(\sigma) - \frac{1}{4}\bar\lambda^I(\partial\!\!\!/\,\sigma^K)\lambda^J\,F_{IJK}\tag{5.67}$$

$$-\frac{i}{8}\,\bar\lambda^I\gamma^{\mu\nu}F^J_{\mu\nu}\,\lambda^K\,F_{IJK} \longrightarrow \begin{cases} -\frac{i}{8}\,\bar\lambda^I\gamma^{mn}F^J_{mn}\,\lambda^K\,F_{IJK} - \frac{i}{4}\,\bar\lambda^I\partial\!\!\!/ b^J\Gamma^{\mathrm{M}}_*\lambda^K\,F_{IJK} & (1,3)\\ -\frac{i}{8}\,\bar\lambda^I\gamma^{mn}F^J_{mn}\,\lambda^K\,F_{IJK} - \frac{e}{4}\,\bar\lambda^I\partial\!\!\!/ b^J\Gamma_*\lambda^K\,F_{IJK} & (0,4)\\ -\frac{i}{8}\,\bar\lambda^I\gamma^{mn}F^J_{mn}\,\lambda^K\,F_{IJK} - \frac{1}{4}\,\bar\lambda^I\partial\!\!\!/ b^J\Gamma^{\mathrm{E}}_*\lambda^K\,F_{IJK} & (0,4)\end{cases}\tag{5.68}$$

The last term in (5.66) does not change its form. To obtain the reduction of the first term, observe that the piece proportional to F_{IJK} vanishes identically by virtue of eq. (A.5). It turns out, however, that this contribution is needed in order to recast the action in a completely (anti-)holomorphic form. The sign of this term has no invariant meaning, as changing it can always be compensated by a spinor rearrangement. We choose '$-$', since in this case the resulting expression on the r.h.s. of (5.67) involves the covariant Dirac operator with respect to the Levi-Civita connection.

We now rewrite the dimensionally reduced terms using the (para-)complex and adapted coordinates. Here we find that the terms containing $\partial\!\!\!/\sigma^I$ and $\partial\!\!\!/ b^I$ can be combined as follows:

$$\begin{aligned}
-\frac{1}{4}\,\bar\lambda^I \partial\!\!\!/\sigma^J\lambda^K\,F_{IJK} - \frac{i}{4}\,\bar\lambda^I \partial\!\!\!/ b^J \Gamma_*^{\mathrm{M}}\lambda^K\,F_{IJK} &= -\frac{1}{4}\,\bar\lambda^I\,(\partial\!\!\!/ F_{IJ})\Gamma_+\lambda^J - \\
&\quad -\frac{1}{4}\,\bar\lambda^I\,(\partial\!\!\!/ \bar F_{IJ})\Gamma_-\lambda^J \quad (1,3)\,, \\
-\frac{1}{4}\,\bar\lambda^I \partial\!\!\!/\sigma^J\lambda^K\,F_{IJK} - \frac{e}{4}\,\bar\lambda^I \partial\!\!\!/ b^J \Gamma_*\lambda^K\,F_{IJK} &= -\frac{1}{4}\,\bar\lambda^I\,(\partial\!\!\!/ F_{IJ})\Gamma_+\lambda^J - \\
&\quad -\frac{1}{4}\,\bar\lambda^I\,(\partial\!\!\!/ \bar F_{IJ})\Gamma_-\lambda^J \quad (0,4)_{PC}\,, \\
-\frac{1}{4}\,\bar\lambda^I \partial\!\!\!/\sigma^J\lambda^K\,F_{IJK} - \frac{1}{4}\,\bar\lambda^I \partial\!\!\!/ b^J \Gamma_*^{\mathrm{E}}\lambda^K\,F_{IJK} &= -\frac{1}{4}\,\bar\lambda^I\,(\partial\!\!\!/ F_{IJ})\Gamma_+^{\mathrm{E}}\lambda^J - \\
&\quad -\frac{1}{4}\,\bar\lambda^I\,(\partial\!\!\!/ \bar F_{IJ})\Gamma_-^{\mathrm{E}}\lambda^J \quad (0,4)_{AC}\,. \qquad (5.69)
\end{aligned}$$

The final form of the fermionic sectors of the dimensional reduced lagrangians it then obtained by decomposing the field strength according to (5.63) and introducing the chiral spinors (5.57). Again the expressions are independent of space-time signature when using (para-)holomorphic coordinates. Combining them with the bosonic terms (5.16) in the minkowskian or in the euclidean case we find the following lagrangian, which applies to both signatures:

$$\begin{aligned}
\mathcal{L}^{d=4} &= -\frac{1}{4}\left(F^I_{-|mn}F_-^{J|mn}\,F_{IJ}(X) + F^I_{+|mn}F_+^{J|mn}\,\bar F_{IJ}(\bar X)\right) - \\
&\quad -\frac{1}{2}\,\partial_m X^I\partial^m \bar X^J\,N_{IJ}(X,\bar X) + Y^I_{ij}Y^{J\,ij}\,N_{IJ}(X,\bar X) - \\
&\quad -\frac{1}{2}\left(\bar\lambda^I_-\partial\!\!\!/\,\lambda^J_+ + \bar\lambda^I_+\partial\!\!\!/\,\lambda^J_-\right)N_{IJ}(X,\bar X) - \\
&\quad -\frac{1}{4}\left(\bar\lambda^I_-\,(\partial\!\!\!/ F_{IJ}(X))\,\lambda^J_+ + \bar\lambda^I_+\,(\partial\!\!\!/ \bar F_{IJ}(\bar X))\,\lambda^J_-\right) - \\
&\quad -\frac{i}{8}\left(\bar\lambda^I_+\gamma^{mn}F^J_{-|mn}\,\lambda^K_+\,F_{IJK} + \bar\lambda^I_-\gamma^{mn}F^J_{+|mn}\,\lambda^K_-\,\bar F_{IJK}\right) - \\
&\quad -\frac{i}{2}\left(\bar\lambda^{iI}_+\lambda^{jJ}_+\,Y^K_{ij}\,F_{IJK} + \bar\lambda^{iI}_-\lambda^{jJ}_-\,\bar Y^K_{ij}\,\bar F_{IJK}\right). \qquad (5.70)
\end{aligned}$$

The supersymmetry variations of this expression are given by (5.64), again for both signatures. We summarize the relations between the minkowskian and euclidean fields in table 1, which together with (5.70) and (5.64) is one of the main results of this paper. For future reference we also give the complete lagrangian of the $(0,4)$ theory in terms of adapted real fields:

$$\begin{aligned}
\mathcal{L}^{(0,4)}_{\text{adapted}} &= -\frac{1}{4}\left(F^{\mathrm{E}\,I}_{-|mn}F_-^{\mathrm{E}\,J|mn}\,F^+_{IJ}(X_+) + F^{\mathrm{E}\,I}_{+|mn}F_+^{\mathrm{E}\,J|mn}\,F^-_{IJ}(X_-)\right) - \\
&\quad -\frac{1}{2}\,\partial_m X^I_+\partial^m X^J_-\,N_{IJ}(X_+,X_-) + Y^I_{ij}Y^{J\,ij}\,N_{IJ}(X_+,X_-) -
\end{aligned}$$

complex coord.	para-complex coord.	adapted coord.
$X^I = \sigma^I + ib^I$	$X^I = \sigma^I + eb^I$	$X_+^I = \sigma^I + b^I$
$\bar{X}^I = \sigma^I - ib^I$	$\bar{X}^I = \sigma^I - eb^I$	$X_-^I = \sigma^I - b^I$
Kähler potential (5.13)	para-Kähler potential (5.43)	para-Kähler potential (5.34)
$F_\pm^I = \frac{1}{2}\left(F^I \pm \frac{1}{i}\tilde{F}^I\right)$	$F_\pm^I = \frac{1}{2}\left(F^I \pm \frac{1}{e}\tilde{F}^I\right)$	$F_\pm^{\mathrm{E}\,I} = \frac{1}{2}\left(F^I \pm \tilde{F}^I\right)$
$\Gamma_\pm = \frac{1}{2}\left(\mathbb{1} \pm \Gamma_*^{\mathrm{M}}\right)$	$\Gamma_\pm = \frac{1}{2}\left(\mathbb{1} \pm \Gamma_*\right)$	$\Gamma_\pm^{\mathrm{E}} = \frac{1}{2}\left(\mathbb{1} \pm \Gamma_*^{\mathrm{E}}\right)$
$\lambda_\pm = \Gamma_\pm\lambda$	$\lambda_\pm = \Gamma_\pm\lambda$	$\xi_\pm = \Gamma_\pm^{\mathrm{E}}\lambda$

Table 1: Summary of the field definitions occurring in the minkowskian and euclidean version of the lagrangian (5.70). For completeness we also summarize the corresponding field definitions in adapted coordinates.

$$
\begin{aligned}
&-\frac{1}{2}\left(\bar{\xi}_-^I \partial\!\!\!/\, \xi_+^J + \bar{\xi}_+^I \partial\!\!\!/\, \xi_-^J\right) N_{IJ}(X_+, X_-) - \\
&-\frac{1}{4}\left(\bar{\xi}_-^I (\partial\!\!\!/ F_{IJ}^+)\xi_+^J + \bar{\xi}_+^I (\partial\!\!\!/ F_{IJ}^-)\xi_-^J\right) - \\
&-\frac{i}{8}\left(\bar{\xi}_+^I \gamma^{mn} F_{-|mn}^{\mathrm{E}\,J} \xi_+^K F_{IJK}^+ + \bar{\xi}_-^I \gamma^{mn} F_{+|mn}^{\mathrm{E}\,J} \xi_-^K F_{IJK}^-\right) - \\
&-\frac{i}{2}\left(\bar{\xi}_+^{iI} \xi_+^{jJ}\, Y_{ij}^K\, F_{IJK}^+ + \bar{\xi}_-^{iI} \xi_-^{jJ}\, Y_{ij}^K\, F_{IJK}^-\right). \qquad (5.71)
\end{aligned}
$$

The supersymmetry variations of this lagrangian are given in (5.65).

Finally we can express the (para-)holomorphic lagrangian using the "new conventions" by substituting $F^{(\mathrm{new})}(X) = \frac{1}{2\hat{\mathrm{i}}} F^{(\mathrm{old})}(X)$ in (5.70), where $\hat{\mathrm{i}} = i$ for minkowskian and $\hat{\mathrm{i}} = e$ for euclidean signature:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{new}}^{d=4} = &-\frac{\hat{\mathrm{i}}}{2}\left(F_{-|mn}^I F_-^{J|mn}\, F_{IJ}(X) - F_{+|mn}^I F_+^{J|mn}\, \bar{F}_{IJ}(\bar{X})\right) - \\
&-\frac{1}{2}\,\partial_m X^I \partial^m \bar{X}^J\, N_{IJ}(X,\bar{X}) + Y_{ij}^I Y^{J\,ij}\, N_{IJ}(X,\bar{X}) - \\
&-\frac{1}{2}\left(\bar{\lambda}_-^I \partial\!\!\!/\, \lambda_+^J + \bar{\lambda}_+^I \partial\!\!\!/\, \lambda_-^J\right) N_{IJ}(X,\bar{X}) - \\
&-\frac{\hat{\mathrm{i}}}{2}\left(\bar{\lambda}_-^I (\partial\!\!\!/ F_{IJ}(X))\lambda_+^J - \bar{\lambda}_+^I (\partial\!\!\!/ \bar{F}_{IJ}(\bar{X}))\lambda_-^J\right) - \\
&-\frac{i\,\hat{\mathrm{i}}}{4}\left(\bar{\lambda}_+^I \gamma^{mn} F_{-|mn}^J\, \lambda_+^K\, F_{IJK} - \bar{\lambda}_-^I \gamma^{mn} F_{+|mn}^J\, \lambda_-^K\, \bar{F}_{IJK}\right) - \\
&-i\,\hat{\mathrm{i}}\left(\bar{\lambda}_+^{iI} \lambda_+^{jJ}\, Y_{ij}^K\, F_{IJK} - \bar{\lambda}_-^{iI} \lambda_-^{jJ}\, \bar{Y}_{ij}^K\, \bar{F}_{IJK}\right). \qquad (5.72)
\end{aligned}
$$

Its supersymmetry variations are given by (5.64).

5.4 Extension to non-cubic prepotentials

We started our construction with a 5-dimensional vector multiplet lagrangian and therefore we obtained 4-dimensional lagrangians with a cubic prepotential with purely real (or, in new conventions, purely imaginary) coefficients. Since every (para-)holomorphic prepotential defines a (para-)holomorphic lagrangian Kähler immersion, all terms in the lagrangian (5.72) maintain their geometric meaning when we allow non-cubic (para-)holomorphic prepotentials. For the euclidean theory this is a consequence of the results of section 2 on special para-Kähler manifolds.

Moreover, the lagrangian is still real. When writing down (5.72), we already anticipated that we wanted to replace the cubic prepotential by a general (para-)holomorphic one. Therefore we systematically used F_{IJK} and $\bar{F}_{IJK}$, despite that in dimensional reduction F_{IJK} is purely real (or, in new conventions, purely imaginary). Note that in our formalism, which uses chiral projections of symplectic Majorana spinors, some of the relative signs between terms are different from those one would get when using chiral projections of Majorana spinors instead, as in [26, 48, 36]. Using the symplectic Majorana condition it is easy to check that the fermionic terms in (5.71) and (5.72) are real. For example, the fifth line of (5.72) is real, for the case of Minkowski signature, because $(\bar{\lambda}_+^I \gamma^{mn} F^J_{-|mn}\, \lambda_+^K\, F_{IJK})^* = -\bar{\lambda}_-^I \gamma^{mn} F^J_{+|mn}\, \lambda_-^K\, \bar{F}_{IJK}$. Note that in the euclidean case one has to check reality with respect to complex and para-complex conjugation separately, because 'reality' with respect to the combined operation would admit expressions proportional to ie, which are not real numbers. But since (5.72) and (5.71) are equal, invariance under para-complex conjugation is guaranteed, so that (5.72) is real if (5.71) is invariant under complex conjugation. In adapted coordinates $F^+(X_+)$, $F^-(X_-)$, $F^{E\,I}_{+|mn}$ and $F^{E\,I}_{-|mn}$ are real. Therefore verifying that the fifth line of (5.71) is real boils down to checking the spinor identity $(\overline{\xi}^i_+ \gamma^{mn} \xi^k_+ \epsilon_{ki})^* = -\overline{\xi}^i_+ \gamma^{mn} \xi^k_+ \epsilon_{ki}$, which easily follows from the symplectic Majorana condition and the properties of γ-matrices under complex conjugation.

If the prepotential is generalized to a *non-cubic* (para-)holomorphic function, then the lagrangian (5.72) is not invariant under the supersymmetry transformations (5.64) any more, because the supersymmetry variations of terms containing the third derivative of the prepotential yield additional terms, which are proportional to the fourth derivative: $\delta F_{IJK}(X) = F_{IJKL}\delta X^L$. Since we want to allow that $F_{IJKL} \neq 0$, we need to add further terms to (5.72). Inspection of the supersymmetry rules (5.64) suggests to add a four-fermion term of the form

$$\Delta\mathcal{L}^{d=4}_{\text{new}} = -\frac{\hat{\mathrm{i}}}{6}\left(F_{IJKL}\bar{\lambda}^{iI}_+\lambda^{jJ}_+\bar{\lambda}^K_{+i}\lambda^L_{+j} - \bar{F}_{IJKL}\bar{\lambda}^{iI}_-\lambda^{jJ}_-\bar{\lambda}^K_{-i}\lambda^L_{-j}\right). \tag{5.73}$$

In order to verify supersymmetry of the combined lagrangian $\mathcal{L}^{d=4}_{\text{new}} + \Delta\mathcal{L}^{d=4}_{\text{new}}$ we use the 4-dimensional Fierz identity (A.12) and the symmetry properties of spinor bilinears. Note that these relations take the same form for both signatures, like the lagrangian and the supersymmetry transformation. It is then straightforward to verify that the variation of the fermions $\lambda^{iI}_\pm$ in $\Delta\mathcal{L}^{d=4}_{\text{new}}$ precisely cancels the terms containing F_{IJKL} and $\bar{F}_{IJKL}$, which appear in the supersymmetry variation of (5.72). Since the five-fermion term, which is generated by the variation of the scalars in $\Delta\mathcal{L}^{d=4}_{\text{new}}$, vanishes identically, the combined lagrangian $\mathcal{L}^{d=4}_{\text{new}} + \Delta\mathcal{L}^{d=4}_{\text{new}}$ is supersymmetric for every (para-)holomorphic prepotential. This proof of supersymmetry is independent of the space-time signature. The only property of the prepotential which enters the calculation is that it is a (para-)holomorphic function, $\partial F/\partial\overline{X}^I = 0$.

For Minkowski signature the general lagrangian for $\mathcal{N}=2$ vector multiplets is of course well known. Its locally supersymmetric version was constructed in [26] using the superconformal tensor calculus.[27] The rigid version of this lagrangian is given explicitly in [27] and [48]. An alternative derivation, based on the rheonomic method was given in [57], see [36] for a summary. We have compared our results, specialized to Minkowski signature, to [26, 27, 48], who also work in special coordinates. Since we use symplectic Majorana spinors, we need to rewrite $\lambda_{\pm}^{iI}$ and $\epsilon_{\pm}^{i}$ in terms of Majorana spinors. This is briefly explained in the appendix, see in particular (A.13). Moreover the auxiliary field Y^{ij} is an SU(2) tensor, and therefore it is also subject to a non-trivial field redefinition. Apart from this one has to take into account different (conventional) normalizations of the fields, which means that our fields differ from the fields use in [27, 48] by constant real factors. When comparing to [26], we also need to rescale the prepotential in order to convert from old to new conventions and we must take the rigid limit of the supergravity lagrangian given in [26]. Taking all these details into account, we find that our supersymmetry transformation rules (5.64) and lagrangian (5.72), (5.73) completely agree with those of [26, 27, 48].

5.5 R-symmetry

In this section we will study the R-symmetry properties of our theories. We focus on the abelian factors $\mathrm{U}(1)_R$ and $\mathrm{SO}(1,1)_R$, which are the additional structure occurring when going from 5 to 4 dimensions. They were already introduced in section 3.3. The main result of this subsection is that the operation of these R-symmetries on the fermions already exhibits the (para-)complex structure on $\mathcal{M}_M$ and $\mathcal{M}_E$, respectively. This implies that the indefiniteness of the scalar kinetic terms is a consequence of implementing the euclidean supersymmetry algebra.[28] We will also show that the general lagrangian is only invariant under the subgroup $\mathbb{Z}_2 \times \mathrm{SU}(2)$ of the R-symmetry group.

So far, we discussed R-symmetry as a property of the supersymmetry algebra. Since the generators $Q_{i\alpha}$ transform the components of a supermultiplet into one another, the fields inherit their behaviour from them. The supersymmetry transformations (5.64) imply that the members of a vector multiplet differ by one unit of R-charge. However, the absolute value of the R-charge is undetermined [75]. Following the literature we take the natural assignment that the fermions λ carry the same R-charge as the supercharges, i.e., charge ± 1. Then the gauge fields carry charge 0, while the scalars carry charge ± 2, see below.

Let us start with the $(0,4)$ theory. The transformation of the supersymmetry generators under $\mathrm{SO}(1,1)_R$ is given by (3.65). Since we take the fermions to carry the same R-charge as the supercharges, this implies that their chiral components transform as

$$\delta\xi_{\pm} = \pm\phi\,\xi_{\pm}\,, \qquad \delta\lambda_{\pm} = \pm e\,\phi\,\lambda_{\pm} \tag{5.74}$$

[27]See also [73, 74] for the construction of the relevant supermultiplets and their transformation rules.

[28]Since one obtains definite scalar kinetic terms in the Osterwalder-Schrader framework [24], it is clear that supersymmetry is implemented in a different way in this approach. We intend to address this in a future publication.

under infinitesimal and

$$\xi_\pm \to \exp(\pm\phi)\,\xi_\pm\,, \qquad \lambda_\pm \to \exp(\pm e\,\phi)\,\lambda_\pm \tag{5.75}$$

under finite R-transformations in the connected component $\mathrm{SO}(1,1)_0$ of the R-symmetry group $\mathrm{SO}(1,1)_R$.[29] This chiral transformation is consistent with the reality condition (3.56).

For the $(1,3)$ theory (3.62) implies

$$\delta\lambda_\pm = \pm i\,\phi\lambda_\pm \;\Rightarrow\; \lambda_\pm \to \exp(\pm i\,\phi)\lambda_\pm\,, \tag{5.76}$$

so that in terms of (para-)holomorphic fields we can write

$$\lambda_+ \to \exp(\hat{\mathrm{i}}\phi)\,\lambda_+, \quad \lambda_- \to \exp(-\hat{\mathrm{i}}\phi)\,\lambda_-\,. \tag{5.77}$$

Thus the compact R-symmetry group $\mathrm{U}(1)_R$ of the Minkowski theory is mapped to a non-compact group $\mathrm{SO}(1,1)_R$ by replacing $i \to e$ in the transformation. Since all the fields are related to the spinors by the supersymmetry variations (5.64), the R-transformations on spinors induce an action of the R-symmetry group on variations of the fields. This action can be integrated to a linear action on the fields, once the scalar fields X^I are specified, which corresponds to a choice of special coordinates on the target manifold. First we observe by looking at the supersymmetry variations of the scalars (5.64), that these also transform under chiral rotations:

$$X^I \to \exp(2\,\hat{\mathrm{i}}\,\phi)\,X^I\,, \qquad \bar{X}^I \to \exp(-2\,\hat{\mathrm{i}}\,\phi)\,\bar{X}^I\,. \tag{5.78}$$

Again we find that (para)-holomorphic fields carry a definite R-charge. The supersymmetry transformations further tell us that the fields $A^I_m, F^I_{\pm|mn}$, and $Y^{ij\,I}$ do not transform.

We can now read off the behaviour of the lagrangian (5.72) under R-symmetry. Since all terms are SU(2)-scalars, invariance with respect to this factor of the R-symmetry group is manifest. With the transformation properties under the abelian factor at hand we see that for a generic choice of the prepotential the $\mathrm{U}(1)_R$ ($\mathrm{SO}(1,1)_R$) is broken to the discrete subgroup $\mathbb{Z}_2$ acting by $\lambda_\pm \to -\lambda_\pm$.

Let us recall that we have fibrewise (para-)complex structures on $T\mathcal{M}$, on spinors (Γ^M_* and Γ_*, respectively) and on two-forms (minus the Hodge star operator). It is remarkable that supersymmetry acts chirally, in the sense that it is consistent with the type decomposition defined by these three (para-)complex structures, i.e., with the decomposition into sections of type $(1,0)$ and $(0,1)$. In fact, supersymmetry relates $X^I \to \lambda^I_+ \to F^I_{-|mn}$ and $\bar{X}^I \to \lambda^I_- \to F^I_{+|mn}$. This ties R-symmetry to the (para-)complex structure of $\mathcal{M}$. We will now show that R-symmetry acts on the fermions by multiplication with the (para-)complex structure of the target manifold.

[29] By exponentiation of infinitesimal transformations we only generate the connected component of $\mathbb{1} \in \mathrm{SO}(1,1)_R$. The transformation in the other connected component take the form $\xi_\pm \to -\exp(\pm\phi)\,\xi_\pm$ and $\lambda_\pm \to +\exp(\pm e\,\phi)\,\lambda_\pm$.

For definiteness, we consider the euclidean theory. We assume that the local immersion ϕ_α of $\mathcal{M}$ into T^*C^N is generic, so that it induces local para-holomorphic coordinates[30] $z^I_\alpha = z^I \circ \phi_\alpha$ and local adapted coordinates $z^I_{+|\alpha} = z^I_+ \circ \phi_\alpha, z^I_{-|\alpha} = z^I_- \circ \phi_\alpha$. For notational convenience we suppress the index α in the following. The spinor fields $\lambda(x)$ are sections of the spinor bundle $\Pi(\mathbb{S}_{SM}) \to \mathbb{R}^{0,4|8}$ over superspace $\mathbb{R}^{0,4|8}$ with even part $E = \mathbb{R}^{0,4}$, which was introduced in subsection 3.4. Using adapted coordinates $z^I_\pm$ on $\mathcal{M}_E$, the anticommuting spinor field λ evaluated at the point $x \in E$ takes the form

$$\lambda(x) = \left(\xi^I_+(x)\frac{\partial}{\partial z^I_+} + \xi^I_-(x)\frac{\partial}{\partial z^I_-}\right)_{|\varphi(x)} \in \Pi\mathbb{S}_{SM} \otimes T_{\varphi(x)}\mathcal{M}_E\,. \tag{5.79}$$

Here $\varphi : E \to \mathcal{M}_E$ is the map which has the scalar fields X^I as its components. In terms of adapted coordinates $z^I_\pm$ we have the decomposition $T_{\varphi(x)}\mathcal{M}_E = T^+_{\varphi(x)}\mathcal{M}_E \oplus T^-_{\varphi(x)}\mathcal{M}_E$ of the tangent space, which can now be identified with $\mathbb{R}^N \oplus \mathbb{R}^N$, equipped with its standard basis, as in example 6 of section 2. Therefore the para-complex structure I acts on $\lambda(x)$ by

$$I\lambda(x) = \left(\xi^I_+(x)\frac{\partial}{\partial z^I_+} - \xi^I_-(x)\frac{\partial}{\partial z^I_-}\right)_{|\varphi(x)}\,, \tag{5.80}$$

which, according to (5.74), is an infinitesimal R-symmetry transformation.

Alternatively we can use para-holomorphic coordinates. This requires to work with the para-complexified tangent space, which has the decomposition $(T_{\varphi(x)}\mathcal{M}_E)^C = T^{(1,0)}_{\varphi(x)}\mathcal{M}_E \oplus T^{(0,1)}_{\varphi(x)}\mathcal{M}_E$. We can identify $(T_{\varphi(x)}\mathcal{M}_E)^C$ with $C^N \oplus C^N$, where both summands are related by para-complex conjugation. Now $\lambda(x)$ takes the form

$$\lambda(x) = \left(\lambda^I_+(x)\frac{\partial}{\partial z^I} + \lambda^I_-(x)\frac{\partial}{\partial \bar{z}^I}\right)_{|\varphi(x)}\,, \tag{5.81}$$

and the para-complex structure acts by

$$I\lambda(x) = \left(e\lambda^I_+(x)\frac{\partial}{\partial z^I} - e\lambda^I_-(x)\frac{\partial}{\partial \bar{z}^I}\right)_{|\varphi(x)}\,, \tag{5.82}$$

which, according to (5.74), again is an infinitesimal R-symmetry transformation.

As discussed in section 2, the para-complex structure generates the group $G = \{\exp(\alpha I) | \alpha \in \mathbb{R}\} \simeq \mathrm{SO}(1,1)_0$. This group acts on the spinors by finite R-symmetry transformations (5.75). We also know from section 2 that, for all points $p \in \mathcal{M}_E$, $\{\exp(\alpha I_p) | \alpha \in \mathbb{R}\}$ is a closed subgroup of the pseudo-orthogonal group $\mathrm{O}(T_p\mathcal{M}_E, g_p)$ defined by the para-Kähler metric g_p at p.[31] Since G is not compact, it follows that the metric of $\mathcal{M}_E$ cannot be definite. Moreover, the eigenspaces of I_p are isotropic and of the same dimension. This shows that the metric is of split type, i.e., $\mathrm{O}(T_p\mathcal{M}_E, g_p) = \mathrm{O}(N,N)$. The integrability of the para-complex structure follows from the description in terms of para-holomorphic coordinates, the para-Kähler condition from the existence of the para-Kähler potential.

[30] The assumption that $z^I, \bar{z}^I$ form a local system of coordinates can be avoided when working with the symplectic vector $(\lambda^I_+, F_{IJ}\lambda^J_+)^T$, which is a tangent vector of the immersed manifold.

[31] This is true although I_p itself is an anti-isometry, and therefore is not an element of the above group.

In the minkowskian theory the discussion is analogous, but this time both the abelian factor of the R-symmetry group and the group generated by the complex structure are compact and isomorphic to U(1). Thus we see that the differences between the scalar geometries of the minkowskian and euclidean theories are rooted in their different R-symmetries. In particular, the fact that the scalar metric of the euclidean theory is indefinite is a consequence of the non-compactness of its R-symmetry group and the consistency of the R-symmetry with the scalar metric.

We also note that the linear action (5.78) of the R-symmetry group on the special coordinates of the scalar manifold coincides with the square of the scalar multiplication (with $\exp(\hat{\mathrm{i}}\phi)$) in $\mathbb{C}^N$ and C^N, respectively. Recall that the linear R-symmetry action on special coordinates is non-canonical from a geometric point of view, because it is coordinate dependent. It is remarkable that it coincides (up to the square) with the (para-)complex scalar multiplication, the latter also being defined by the choice of (para-)holomorphic coordinates. The link between the R-symmetry operation on the coordinates X^I and the (para-)complex structure, as an endomorphism of the tangent bundle of the target, is again provided by supersymmetry, through the variation (5.60), which ties δX^I to λ^I_+. It may be surprising that although X^I is a coordinate and not a tangent vector, it transforms linearly under R-symmetry. The reason is that X^I is a special coordinate, which sits in the same supermultiplet as the tangent vector λ^I_+.

Finally we remark that our results fix the ambiguities which occur when one tries to apply the $i \to e$ substitution rule of [5] naively. As we have seen this rule means that one has to replace the complex structure of the scalar manifold by a para-complex structure. However, the lagrangian also contains factors of i which have a different origin, namely the complex structure of the spinor module. Such factors of i remain unchanged. Also note that the fields $\lambda(x)_\pm$ of the euclidean theory are complex as spinors, but para-complex as tangent vectors. This explains the geometric meaning of expressions like $e\gamma^m$, where the para-complex unit is multiplied with a complex matrix: $e\gamma^m$ acts as multiplication by e on $(T_{\varphi(x)}\mathcal{M}_E)^C$ and as Clifford multiplication on $\mathbb{S}_{SM}$.

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A. Notations and conventions

In this paper we follow the conventions of refs. [40, 43]. Most of the relevant details are given in section 3, where we also explain the relation to [41], which treats supersymmetry from the mathematicians point of view. Here we collect several further formulae, which are needed for the calculations in sections 4 and 5.

Indices	Description	Range
$\mu, \nu, \ldots$	space-time indices in dimension (1,4)	0,1,2,3,5
$m, n, \ldots$	space indices in dimension (0,4)	1,2,3,5
	space-time indices in dimension (1,3)	0,1,2,3
$i, j, \ldots$	USp(2) indices	1,2
$I, J, \ldots$	vector multiplet labels	1,. . . ,N
$\alpha, \beta, \ldots$	spinor indices	1,2,3,4

Table 2: Summary of our index conventions.

Spinor bilinears and γ matrix identities

We summarize our conventions for indices in table 2. In the following λ^i, χ^i are anticommuting (Grassmann-valued) symplectic Majorana spinors in dimension (1,4), see section 3.4. They satisfy (3.17)

$$(\lambda^i)^* = -B\lambda^j \epsilon_{ji} \,. \tag{A.1}$$

The matrix B is defined in section 3 and satisfies $BB^* = -\mathbb{1}$. Here, ϵ_{ij} is an antisymmetric two-by-two matrix, which we take to be (3.18)

$$(\epsilon_{ij}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.2}$$

The indices $i, j, \ldots = 1, 2$ are raised and lowered according to the so-called NW-SE convention: $\lambda_i := \lambda^j \epsilon_{ji}$ and $\lambda^i = \epsilon^{ij}\lambda_j$, where $\epsilon^{ij} = \epsilon_{ij}$ and therefore $\epsilon^{ik}\epsilon_{kj} = -\delta^i_j$.

We now list useful spinor identities valid in dimension (1,4). Note that these can be re-interpreted in dimensions (1,3) and (0,4), as discussed in section 3.

We define

$$\gamma^{(p)} = \gamma^{\mu_1 \cdots \mu_p} = \gamma^{[\mu_1}\gamma^{\mu_2} \cdots \gamma^{\mu_p]} = \frac{1}{p!}\left(\gamma^{\mu_1} \ldots \gamma^{\mu_p} \pm \text{cyclic}\right). \tag{A.3}$$

Changing the order of spinors in a bilinear leads to the following signs:

$$\bar{\lambda}^i \gamma_{(p)} \chi^j = t_p \, \bar{\chi}^j \gamma_{(p)} \lambda^i \,, \qquad \begin{cases} t_p = -1 & \text{for} \quad p = 2, 3 \\ t_p = +1 & \text{for} \quad p = 0, 1 \,. \end{cases} \tag{A.4}$$

In particular this implies the useful identities

$$\bar{\lambda}^{i(I}\lambda_i{}^{J)} = 0 \,, \qquad \bar{\lambda}^{i(I}\gamma^\mu \lambda_i{}^{J)} = 0 \,. \tag{A.5}$$

In order to check that the $(1, 4)$ dimensional lagrangian is supersymmetric, we made use of the following Fierz rearrangements:

$$\begin{aligned} \lambda_j \bar{\eta}^i &= -\frac{1}{4}\,\bar{\eta}^i\lambda_j - \frac{1}{4}\,\gamma_\mu\left(\bar{\eta}^i\gamma^\mu\lambda_j\right) + \frac{1}{8}\,\gamma_{\mu\nu}\left(\bar{\eta}^i\gamma^{\mu\nu}\lambda_j\right), \\ \bar{\lambda}^{[i}\eta^{j]} &= -\frac{1}{2}\,\bar{\lambda}\eta\,\epsilon^{ij} \,. \end{aligned} \tag{A.6}$$

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Using the Clifford algebra, the following identities can be obtained:

$$\gamma^{(q)}\gamma_{(p)}\gamma_{(q)} = c_{p,q}\gamma_{(p)}\,,$$

$c_{p,q}$	$q=1$	$q=2$
$p=0$	5	-20
$p=1$	-3	-4
$p=2$	1	4
$p=3$	1	4

(A.7)

The totally antisymmetric ϵ-symbols are defined as

$$\begin{aligned} \epsilon_{01235} &= -1 = -\epsilon^{01235}\,, \qquad (1,4) \\ \epsilon_{0123} &= 1 = -\epsilon^{0123}\,, \qquad (1,3) \\ \epsilon_{1235} &= 1 = +\epsilon^{1235}\,, \qquad (0,4)\,. \end{aligned} \tag{A.8}$$

These satisfy the following contraction identity ($t+s=n$):

$$\epsilon_{\rho_1\dots\rho_p\mu_1\dots\mu_q}\,\epsilon^{\rho_1\dots\rho_p\nu_1\dots\nu_q} = (-1)^t\,p!q!\,\delta^{[\nu_1}_{[\mu_1}\dots\delta^{\nu_q]}_{\mu_q]}\,. \tag{A.9}$$

In even dimensions, we can relate $\gamma_{(p)}$ to $\gamma^{(n-p)}$ by the following identity

$$\gamma_{\mu_1\dots\mu_p} = \frac{i^{n/2+t}}{(n-p)!}\,\epsilon_{\mu_1\dots\mu_n}\,\Gamma_*\,\gamma^{\mu_n\dots\mu_{p+1}}\,. \tag{A.10}$$

Here, Γ_* is proportional to the product of all gamma matrices

$$\Gamma_* = (-i)^{n/2+t}\,\gamma_0\dots\gamma_{n-1}\,, \qquad \Gamma_*\Gamma_* = \mathbb{1}\,. \tag{A.11}$$

Note that Γ_* defined here is $-\Gamma_*^{\mathrm{E}}$ as used in the main part of the paper.

Verification of supersymmetry in 4 dimensions

To verify supersymmetry for a general (para-)holomorphic prepotential in 4 dimensions, we need to use the symmetry properties of spinor bilinears and a 4-dimensional Fierz identity. Both follow from the corresponding 5-dimensional expressions by dimensional reduction (using the formulae derived in sections 3.3 and 5.2). From the 5-dimensional Fierz identity we obtain 4-dimensional Fierz identity

$$\begin{aligned} \lambda^j_\pm\overline{\eta}^i_\pm &= -\frac{1}{2}(\overline{\eta}^i_\pm\lambda^j_\pm)\Gamma_\pm + \frac{1}{8}(\overline{\eta}^i_\pm\gamma^{mn}\lambda^j_\pm)\gamma_{mn}\,, \\ \lambda^j_\mp\overline{\eta}^i_\pm &= -\frac{1}{2}(\overline{\eta}^i_\pm\gamma^m\lambda^j_\mp)\gamma_m\Gamma_\mp\,, \end{aligned} \tag{A.12}$$

which takes the same form for both signatures ($\Gamma_\pm$ are the projectors adapted to the (para-)holomorphic parametrization, see section 5.2). Moreover, the symmetry properties of spinor bilinears are the same as in 5 dimensions. This shows that all formulae needed to verify supersymmetry in 4 dimensions can be written in a form which is independent of the signature.

Majorana and symplectic Majorana spinors in dimension (1,3)

While we use symplectic Majorana spinors in this paper, most of the literature on $\mathcal{N}=2$ supersymmetry in dimension (1,3) uses Majorana spinors. Therefore we review how both formulations are related. In 5 dimensions $\sigma=-1$ and $\tau=1$ is the only consistent choice in (3.10). But in $n=(1,3)$ one can find matrices C_+ and C_- which satisfy (3.10) with $\sigma(\mathsf{C}_+)=-1$, $\tau(\mathsf{C}_+)=-1$ and $\sigma(\mathsf{C}_-)=-1$, $\tau(\mathsf{C}_-)=+1$. They are related by $\mathsf{C}_+=\mathsf{C}_-\gamma_5$. Only the representation C_- can consistently be "lifted" to the 5-dimensional Clifford algebra. Using $B_\pm^{\mathrm{T}} \equiv \mathsf{C}_\pm A^{-1}$ one finds $B_+^* B_+ = \mathbb{1}$, while $B_-^* B_- = -\mathbb{1}$. Hence the "+"-representation admits Majorana spinors, $\Omega^* = B_+\Omega$, while the "$-$"-representation allows symplectic Majorana spinors, $(\lambda^i)^* = \epsilon_{ij} B_- \lambda^j$, only. A symplectic Majorana spinor $(\lambda^1,\lambda^2)^T$ can be written in terms of Majorana spinors $\Omega^{(1)},\Omega^{(2)}$ as

$$\begin{aligned}\lambda^1 &= \Omega^{(1)} - i\Omega^{(2)}\\ \lambda^2 &= -B_-^*(\Omega^{(1)} + i\Omega^{(2)})\,.\end{aligned} \tag{A.13}$$

This formula has been obtained by using the relation (3.20) between symplectic Majorana and Dirac spinors and then decomposing the Dirac spinor into two Majorana spinors. The transformation (A.13) is not the most general one: it is possible to rescale the $\Omega^{(i)}$ by an overall real factor and to apply a real rotation to the vector $(\Omega^{(1)},\Omega^{(2)})^T$. In fact, in order to relate our results to those of [26, 27, 48], one needs to use both a rotation and a rescaling.

From (A.13) one easily derives the relation between the chirally projected spinors $\lambda^i_\pm$ and $\Omega^{(i)}_\pm$. Note that in the conventions of [26, 27, 48] the chirality is encoded in the position of the SU(2) index. They denote the right- and left-handed projections of $\Omega^{(i)}$ by Ω_i and Ω^i, respectively.

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